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AN OSCILLATION CRITERION FOR SECOND ORDER NONLINEAR DIFFERENTIAL EQUATION

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A new oscillation criterion is given for the second order nonlinear differential equation

$$\ddot{x}(t) + q(t) f(x(t)) = 0$$

where the coefficient $q(t)$ is not assumed to be nonnegative for all large values of t. Condition on f of the form $\int^{\pm\infty} \frac{du}{f(u)} < \infty$, used by Onose, Philos and Wong is discarded.

1. INTRODUCTION

Consider the second order nonlinear differential equation

$$\ddot{x}(t) + q(t) f(x(t)) = 0, \left(\cdot = \frac{d}{dt} \right) \qquad \text{...(1)}$$

where $q : [t_0, \infty) \to R = (-\infty, \infty)$, $f : R \to R$ are continuous and $x f(x) > 0$ for $x \neq 0$.

We consider only those solutions of eqn. (1) which exist on $[t_0, \infty)$. A solution of eqn. (1) is said to be oscillatory if it has arbitrarily large zeros, and otherwise it is said to be nonoscillatory. Equation (1) is called oscillatory if all such solutions are oscillatory.

Recently, Philos[2] considered the strongly superlinear equation of the form (1) i.e., the function f in eqn. (1) is such that

$$\int^{+\infty} \frac{du}{f(u)} < \infty \text{ and } \int^{-\infty} \frac{du}{f(u)} < \infty \qquad \text{...(2)}$$

and proved the following oscillation criterion.

Theorem A—Suppose that condition (2) holds,

$$f'(x) \geq k > 0 \text{ for } x \neq 0, \left(' = \frac{d}{dx} \right) \qquad \text{...(3)}$$

and let ρ be a positive continuously differentiable function on the interval $[t_0, \infty)$ such that $\dot{\rho}$ is nonnegative and decreasing on $[t_0, \infty)$. Equation (1) is oscillatory if

$$\liminf_{t\to\infty} \int_{t_0}^{t} \rho(s)\, q(s)\, ds > -\infty \qquad \text{...(4)}$$

and

$$\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} \rho(u)\, q(u)\, du\, ds = \infty. \qquad \text{...(5)}$$

His criterion extended and improved some of the results due to Onose[1] and Wong[3-5]. Theorem A is only concerned with oscillatory behaviour of strongly superlinear equations of the form of eqn. (1). Therefore, the purpose of this paper is to establish a new oscillation criterion for eqn. (1), where condition (2) is discarded. We also discuss the asymptotic behaviour of the forced equation

$$\ddot{x}(t) + q(t) f(x(t)) = e(t) \qquad \text{...(6)}$$

where $e : [t_0, \infty) \to R$ is continuous.

Our results extend and improve some of the results of Onose[1], Philos[2] and Wong[3-5].

2. Main Results

Theorem 1—Let condition (3) hold and suppose that there is a differentiable function

$$\rho : [t_0, \infty) \to (0, \infty)$$

such that

$$\dot{\rho}(t) \geqslant 0 \text{ for } t \geqslant t_0 \text{ and } \limsup_{t\to\infty} \frac{1}{t^2} \int_{t_0}^{t} \rho(s)\, ds < \infty \qquad \text{...(7)}$$

$$\int^{\infty} \frac{1}{\rho(s)}\, ds = \infty \qquad \text{...(8)}$$

$$\liminf_{t\to\infty} \int_{t_0}^{t} \left[\rho(s)\, q(s) - \frac{\dot{\rho}^2(u)}{4k\, \rho(s)}\right] ds > -\infty \qquad \text{...(9)}$$

and

$$\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} \left[\rho(u)\, q(u) - \frac{\dot{\rho}^2(u)}{4\, k\rho(u)}\right] du\, ds = \infty. \qquad \text{...(10)}$$

Then every solution of eqn. (1) is oscillatory.

PROOF : Let $x(t)$ be a nonoscillatory solution of eqn. (1). Without loss of generality, we assume that $x(t) \neq 0$ for all $t \geqslant t_0$. Furthermore, we assume that $x(t) > 0$ for $t \geqslant t_0$, since the substitution $u = -x$ transforms eqn. (1) into an equation of the same form subject to the assumptions of the theorem.

Now we define

$$w(t) = \rho(t) \frac{\dot{x}(t)}{f(x(t))} \text{ for every } t \geqslant t_0.$$

Then for all $t \geqslant t_0$ we obtain

$$\dot{w}(t) = -\rho(t) q(t) + \frac{\dot{\rho}(t)}{\rho(t)} w(t) - \frac{1}{\rho(t)} f'(x(t)) w^2(t)$$

$$\leqslant -[\rho(t) q(t) - \frac{\dot{\rho}^2(t)}{4k\rho(t)}] - \frac{1}{\rho(t)}[\sqrt{k}\, w(t) - \frac{\dot{\rho}(t)}{2\sqrt{k}}]^2. \quad \ldots(11)$$

Hence, for all $t \geqslant t_0$ we have

$$w(t) \leqslant w(t_0) - \int_{t_0}^{t} [\rho(s) q(s) - \frac{\dot{\rho}^2(s)}{4k\rho(s)}] ds - \int_{t_0}^{t} \frac{1}{\rho(s)}[\sqrt{k}\, w(s) - \frac{\dot{\rho}(s)}{2\sqrt{k}}]^2 ds. \quad \ldots(12)$$

Next, we consider the following three cases for the behaviour of $\dot{x}$:

Case 1—$\dot{x}$ is oscillatory. Then there exists a sequence $\{t_m\}_{m=1,2\ldots}$ in $[t_0, \infty)$ with $\lim_{m\to\infty} t_m = \infty$ and such that $\dot{x}(t_m) = 0$ $(m = 1,2,\ldots)$. Thus, (12) gives

$$\int_{t_0}^{t_m} \frac{1}{\rho(s)} [\sqrt{k}\, w(s) - \frac{\dot{\rho}(s)}{2\sqrt{k}}]^2 ds \leqslant w(t_0) - \int_{t_0}^{t_m} [\rho(s) q(s) - \frac{\dot{\rho}^2(s)}{4k\,\rho(s)}]^2 ds, \quad m = 1, 2, \ldots$$

and hence, by (g) we conclude that

$$\int_{t_0}^{\infty} \frac{1}{\rho(s)}[\sqrt{k}\, w(s) - \frac{\dot{\rho}(s)}{2\sqrt{k}}]^2 ds < \infty.$$

So, for some positive K we have

$$\int_{t_0}^{\infty} \frac{1}{\rho(s)} [k\, w(s) - \frac{\dot{\rho}(s)}{2\sqrt{k}}]^2 ds \leqslant K \text{ for every } t \geqslant t_0.$$

By the Schwarz inequality, for $t \geqslant t_0$ we get

$$\left| -\int_{t_0}^{t} [\sqrt{k}\, w(s) - \frac{\dot{\rho}(s)}{2\sqrt{k}}]\, ds \right|^2 = \left| \int_{t_0}^{t} \sqrt{\rho(s)}\left[\frac{1}{\sqrt{\rho(s)}}(\sqrt{k}\, w(s) - \frac{\dot{\rho}(s)}{2\sqrt{k}})\right] ds \right|^2 \leqslant \left[\int_{t_0}^{t} \rho(s)\, ds\right] \int_{t_0}^{t} \frac{1}{\rho(s)} [\sqrt{k}\, w(s) - \frac{\dot{\rho}(s)}{2\sqrt{k}}]^2\, ds$$

$$\leqslant K \int_{t_0}^{t} \rho(s)\, ds.$$

From (7), there exists a positive constant L such that

$$\int_{t_0}^{t} \rho(s)\, ds \leqslant Lt^2 \text{ for every } t \geqslant t_0$$

and hence for all $t \geqslant t_0$ we obtain

$$-\int_{t_0}^{t} [\sqrt{k}\, w(s) - \frac{\dot{\rho}(s)}{2\sqrt{k}}]\, ds \leqslant \sqrt{KL}\; t.$$

Since $\dot{\rho}(t) \geqslant 0$ for $t \geqslant t_0$ we have

$$-\int_{t_0}^{t} w(s)\, ds \leqslant \sqrt{\frac{KL}{k}}\; t.$$

Furthermore, (12) gives

$$\int_{t_0}^{t} [\rho(s)\, q(s) - \frac{\dot{\rho}^2(s)}{4k\rho(s)}]\, ds \leqslant -w(t) + w(t_0)$$

and therefore, for all $t \geqslant t_0$

$$\int_{t_0}^{t}\int_{t_0}^{s} [\rho(u)\, q(u) - \frac{\dot{\rho}^2(u)}{4k\rho(u)}]\, du\, ds \leqslant -\int_{t_0}^{t} w(s)\, ds + (t - t_0)\, w(t_0)$$

$$\leqslant \sqrt{\frac{KL}{k}}\; t + (t - t_0)\, w(t_0)$$

and hence

$$\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} [\rho(u)\, q(u) - \frac{\dot{\rho}^2(u)}{4k\rho(u)}]\, du\, ds \leqslant \sqrt{\frac{KL}{k}} + \left(1 - \frac{t_0}{t}\right) w(t_0).$$

This contradicts condition (10). Thus $\dot{x}(t)$ is of constant sign for all $t \geqslant t_0$.

Case 2—$\dot{x} > 0$ on $[t_1, \infty)$ for some $t_1 \geqslant t_0$. In this case, from (12) it follows that for $t \geqslant t_1$

$$\int_{t_0}^{t} [\rho(s)\, q(s) - \frac{\dot{\rho}^2(s)}{4k\rho(s)}]\, ds \leqslant w(t_0)$$

and consequently

$$\frac{1}{t}\int_{t_1}^{t}\int_{t_0}^{s} [\rho(u)\, q(u) - \frac{\dot{\rho}^2(u)}{4k\rho(u)}]\, du\, ds \leqslant (1 - \frac{t_1}{t})\, w(t_0)$$

which again contradicts condition (10).

Case 3—$\dot{x} < 0$ on $[t_1, \infty)$ for some $t_1 \geqslant t_0$. From (11) and the fact that ρ is nondecreasing on $[t_0, \infty)$ it follows that

$$- w(t) \geqslant - w(t_1) + \int_{t_1}^{t} [\rho(s)\, q(s) - \frac{\dot{\rho}^2(s)}{4k\rho(s)}]\, ds + \int_{t_1}^{t} \frac{1}{\rho(s)} f'(x(s))\, w^2(s)\, ds. \qquad ...(13)$$

If

$$\int_{t_1}^{\infty} \frac{1}{\rho(s)} f'(x(s))\, w^2(s)\, ds < \infty$$

then condition (3) ensures that

$$\int_{t_1}^{\infty} \frac{1}{\rho(s)} w^2(s)\, ds < \infty$$

and hence we can arrive at a contradiction by the procedure of Case 1. So, we suppose that the above improper integral is infinite. From (13) and condition (9) we derive

$$- w(t) \geqslant C + \int_{t_1}^{t} \frac{1}{\rho(s)}\, w^2(s)\, f'(x(s))\, ds$$

where C is a constant. Furthermore we choose a $t_2 \geqslant t_1$ so that

$$C + \int_{t_1}^{t_2} \frac{1}{\rho(s)}\, w^2(s)\, f'(x(s))\, ds = C_1 > 0$$

and then for every $t \geqslant t_2$ we get

$$\frac{1}{\rho(t)} w^2(t) f'(x(t)) [C + \int_{t_1}^{t} \frac{1}{\rho(s)} w^2(s) f'(x)(s)) ds]^{-1}$$
$$\geqslant - \frac{\dot{x}(t) f'(x(t))}{f(x(t))}$$

and hence, by integrating over $[t_2, t]$, we obtain

$$\log \frac{1}{C_1} [C + \int_{t_1}^{t} \frac{1}{\rho(s)} w^2(s) f'(x)) ds] \geqslant \log \frac{f(x(t_2))}{f(x(t))}.$$

Thus

$$C + \int_{t_1}^{t} \frac{1}{\rho(s)} w^2(s) f'(x(s)) ds \geqslant C_2 \frac{1}{f(x(t))}, \text{ for all } t \geqslant t_2$$

where $C_2 = C_1 f(x(t_2)) > 0$. So (13) yields

$$\dot{x}(t) \leqslant - C_2 \frac{1}{\rho(t)} \text{ for every } t \geqslant t_2$$

and consequently we have

$$x(t) \leqslant x(t_2) - C_2 \int_{t_2}^{t} \frac{1}{\rho(s)} ds \to -\infty \text{ as } t \to \infty$$

a contradiction to the fact that $x(t) > 0$ for $t \geqslant t_0$. This completes the proof.

Theorem 2—Let conditions (7) and (8) in Theorem 1 be replaced by

$$\dot{\rho}(t) \geqslant 0 \text{ and } \ddot{\rho}(t) \leqslant 0 \text{ for } t \geqslant t_0. \qquad ...(14)$$

Then the conclusion of Theorem 1 holds.

PROOF : The proof is similar to that of Theorem 1 and hence is omitted.

From the proof of Theorem 1 we see the following results hold :

Corollary 1—Let the conditions (3), (7) and (8) hold and

$$\int^{\infty} \frac{\dot{\rho}^2(s)}{\rho(s)} ds < \infty. \qquad ...(15)$$

Equation (1) is oscillatory if condition (4) and (5) hold.

Corollary 2—Let conditions (3), (4), (5), (14) and (15) hold, then equation (1) is oscillatory.

Remark 1 : Theorem 1 includes Theorem 3 in Philos[2] as a special case, and since condition (2) is disregarded in our criterion, Corollary 1 extends and improves Theorem A when condition (15) holds. On the other hand, Theorem 1 can be applied in some cases in which Theorem A is not applicable. Such a case is described in the following example.

Example 1—Consider the differential equation

$$\ddot{x}(t) + (-t^{-1/2} \sin t + \tfrac{1}{2} t^{-3/2} (2 + \cos t))\, f(x(t)) = 0,\ t \geqslant t_0 = \frac{\pi}{2} \qquad \ldots(16)$$

where f is any one of the following functions :

(i) $f(x) = cx,\ x \in R$ for $c > 0$.

(ii) $f(x) = |x|^{\gamma} \operatorname{sgn} x + kx,\ x \in R$ for $\gamma > 0$ and $k > 0$.

(iii) $f(x) = x \log^2 (\mu + |x|),\ x \in R$ for $\mu > 1$.

(iv) $f(x) = x e^{\lambda |x|},\ x \in R,\ \lambda \geqslant 0$.

We let $\rho(t) = t$. Conditions (3), (7) and (8) are easy to verify and

$$\int_{t_0}^{t} [\rho(s)\, q(s) - \frac{\dot{\rho}^2(s)}{4k\rho(s)}]\, ds = \int_{\pi/2}^{t} \left[\frac{2 + \cos s}{2\sqrt{s}} - \sqrt{s} \sin s - \frac{1}{4k\, s} \right] ds$$

$$= \sqrt{t}\,(2 + \cos t) - 2\left(\frac{\pi}{2}\right)^{1/2} - \frac{1}{4k} \ln \frac{2t}{\pi} > \sqrt{t} - 2\left(\frac{\pi}{2}\right)^{1/2} - \frac{1}{4k} \ln \frac{2t}{\pi}$$

where k is as in condition (3), and

$$\frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} [\rho(u)\, q(u) - \frac{\dot{\rho}^2(u)}{4k\rho(u)}]\, du\, ds > \tfrac{1}{2} \int_{\pi/2}^{t} [\sqrt{s} - 2\left(\frac{\pi}{2}\right)^{1/2} - \frac{1}{4k} \ln \frac{2s}{\pi}]\, ds = \frac{2}{3}\sqrt{t} + \frac{4}{3}\left(\frac{\pi}{2}\right)^{3/2} - 2\left(\frac{\pi}{2}\right)^{1/2} - \frac{1}{4k} \ln \frac{2t}{\pi} + \frac{1}{4k}\left(1 - \frac{\pi}{2t}\right).$$

Conditions (9) and (10) of Theorem 1 are satisfied, and hence every solution of eqn. (16) is oscillatory. We may note that Theorem A can be applied to eqn. (16) when f is

strongly superlinear i. e. for the case (ii) with $\gamma > 1$, case (iii) and case (iv) with $\lambda > 0$. Also, Theorem 3 in Philos[2] is not applicable here since

$$\limsup_{t\to\infty} \frac{1}{t} \int_{\pi/2}^{t} \int_{\pi/2}^{s} \left[\frac{2 + \cos u}{2u\sqrt{u}} - \frac{\sin u}{\sqrt{u}}\right] du\, ds < \infty .$$

We conclude that the class of the function f described is larger than that in Philos[2].

The following theorem is concerned with the asymptotic behaviour of all solutions of the forced equation (6)

Theorem 3—In addition to the hypotheses of Theorem 1, we let

$$\int^{\infty} \rho(s) \mid e(s) \mid ds < \infty \qquad \text{...(17)}$$

then $\liminf_{t\to\infty} \mid x(t) \mid = 0$ for all solutions x of eqn. (6).

PROOF : Let x be a solution of eqn. (6) on $[t_0, \infty)$ with $\liminf_{\to\infty} \mid x(t) \mid > 0$. Clearly x is nonoscillatory. Without loss of generality, we assume that $x(t) > 0$ for $t \geqslant t_0$. Furthermore, we consider the function W defined as in the proof of Theorem1 and then for $t \geqslant t_0$ we get

$$\dot{w}(t) = \dot{\rho}(t) \frac{\dot{x}(t)}{f(x((t)))} \rho -(t)\, q(t) + \frac{1}{f(x)(t))} \rho(t)\, e(t) - \frac{1}{\rho(t)}$$

$$w^2(t) f'(x(t)) \leqslant -\rho(t)\, q(t) + \frac{1}{f(c)} \rho(t) \mid e(t) \mid$$

$$+ \dot{\rho}(t) \frac{\dot{x}(t)}{f(x(t))} - \frac{1}{\rho(t)} w^2(t) f'(x(t))$$

where $c = \inf_{t \geqslant t_0} x(t) > 0$. Thus

$$\int_{t_0}^{t} \rho(s)\, q(s)\, ds \leqslant -w(t) + w(t_0) + \int_{t_0}^{t} \frac{\dot{\rho}(s)}{\rho(s)} w(s)\, ds$$

$$+ \frac{1}{f(c)} \int_{t_0}^{t} \rho(s) \mid e(s) \mid ds - \int_{t_0}^{t} \frac{1}{\rho(s)} w^2(s) f'(x(s))\, ds.$$

$$\leqslant M - w(t) + \int_{t_0}^{t} \frac{\dot{\rho}(s)}{\rho(s)} w(s)\, ds - \int_{t_0}^{t} \frac{1}{\rho(s)} w^2(s) f'(x(s))\, ds.$$

where $M = w(t_0) + \dfrac{1}{f(c)} \displaystyle\int_{t_0}^{\infty} \rho(s) \mid e(s) \mid ds$. Now, we can complete the proof by procedure of the proof of Theorem 1 and hence we omit the detail.

The following examples are illustrative.

Example 2—Consider the second order forced equation

$$\ddot{x}(t) + \left[\frac{2 + \cos t}{2t\sqrt{t}} - \frac{\sin t}{\sqrt{t}}\right] x(t) = \frac{15}{4t^{7/2}} - \frac{1}{t_2} \sin t$$

$$+ \frac{2 + \cos t}{2t^3},\ t \geqslant t_0 = \frac{\pi}{2}. \qquad \ldots(18)$$

All conditions of Theorem 3 are satisfied with $\rho(t) = t$. Hence, all solutions x of equation (18) satisfy $\liminf_{t \to \infty} \mid x(t) \mid = 0$. One such solution is $x(t) = 1/t\sqrt{t}$. We may note that Proposition 1 in Philos[2] is not applicable to equation (18).

Example 3—The equation

$$\ddot{x}(t) + x(t) = \frac{2 \sin t}{t_3} - \frac{2}{t^2} \cos t,\ t > 0 \qquad \ldots(19)$$

has the oscillatory solution $x(t) = \dfrac{\sin t}{t}$. All conditions of Theorem 3 are satisfied with $\rho(t) = 1$, $t \geqslant t_0 > 0$.

Remark 2 : From Example 2, we see that eqn. (18) without forcing term is oscillatory by Theorem 1 while the forced equation (18) has a nonoscillatory solution $x(t) = \dfrac{1}{t} \to 0$ as $t \to \infty$. On the other hand, from Example 3, we find that the unforced equation $\ddot{x} + x = 0$ is oscillatory and has solution $\sin t$ and $\cos t \not\to 0$ as $t \to \infty$ and eqn. (19) has oscillatory solution $x(t) = \sin t/t \to 0$ as $t \to \infty$. Therefore, it seems that the size of the forcing term is responsible for generating such behaviours. Now, it remains an open problem to find conditions on the forcing term "e" in eqn. (6) so that the oscillatory character of the unforced equation is either to be changed or maintained.

Remark 3 : The results of this paper can be extended to more general equations of the form

$$(a(t)\,\dot{x}(t))^{\cdot} + p(t)\,\dot{x}(t) + q(t) f(x(t)) = 0 \qquad \ldots(20)$$

and

$$(a(t)\,\dot{x}(t))^{\cdot} + p(t)\,\dot{x}(t) + q(t)\ f(x(t)) = e(t) \qquad \ldots(21)$$

where $a, e, p, q : [t_0, \infty) \to R$ and $f : R \to R$ are continuous, $a(t) > 0$ for $t \geqslant t_0$ and $x f(x) > 0$ for $x \neq 0$.

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ON THE UNIFORM STABILITY OF A SYSTEM OF DIFFERENTIAL EQUATIONS WITH COMPLEX COEFFICIENTS

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In a recent paper[4], necessary and sufficient conditions for the asymptotic stability of a system of differential equations of dimension at most 4 with complex coefficients were established. In the present work, other types of stability are considered, the results of Zahreddine and Elshehawey[4] are extended, and necessary and sufficient condition for the stability and uniform stability of the above systems are established.

1. Introduction

Consider the homogeneous, first order linear system of ordinary differential equations of n-dimensions $X' = AX$ where A is an $n \times n$ complex constant matrix and X (t) is a column vector of the n dependent variables. The characteristic equation is a polynomial equation of degree n whose roots are the eigenvalues of A.

We follow the definitions of asymptotic stability, uniform stability and stability as given in Jordan and Smith[2], and we note that when A is constant, then stability implies uniform stability [Jordan and Smith[2], remarks following definition (9.3)].

By (Jordan and Smith[2], Theorem 9.3), the question of stability of the system $X' = AX$ is related to the nature of the eigenvalues of A, when A is complex, more details may be found in Boyce and Diprima[1]. With the help of Hurwitz polynomials and positive functions, we were able to establish (Section 3 of Zahreddine and Elshehawey[4]) necessary and sufficient conditions for the asymptotic stability of the system $X' = AX$ where A is a complex matrix of dimension at most 4. We intend to include not only asymptotic stability, but also stability and therefore uniform stability. We extend the results of Zahreddine and Elshehawey[4] and establish necessary and sufficient conditions for the stability and uniform stability of the systems described above.

Before we proceed, we recall some basic definitions and facts.

2. Definitions and Notations

Definition 2.1—The polynomial $f(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \ldots + a_{n-1} \lambda + a_n$ with complex coefficients is a Hurwitz polynomial if all its roots have negative real parts.

Definition 2.2—If $g(\lambda)$ is a rational function, its paraconjugate is defined by $g^*(\lambda) = \overline{g(-\bar{\lambda})}$, where $\bar{\lambda}$ denotes the complex conjugate of λ.

When $f(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \ldots\ldots + a_{n-2}\lambda^2 + a_{n-1}\lambda + a_n$

then

$$f^*(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} \bar{a}_1 \lambda^{n-1} + \ldots + \bar{a}_{n-2}\lambda^2 - \bar{a}_{n-1}\lambda + \bar{a}_n.$$

Definition 2.3—A function $h(\lambda)$ is said to be positive if $\operatorname{Re} h(\lambda) > 0$ whenever $\operatorname{Re}\lambda > 0$.

The study of Hurwitz polynomials may be reduced to that of positive functions Levinson and Redheffer[3], (Theorem 5.1), and, by Theorem 5.2 of Levinson and Redheffer[3] any rational function h such that h and $-h^*$ are both positive can be written in the form :

$$h(\lambda) = a + b\lambda + \frac{b_1}{\lambda - iw_1} + \frac{b_2}{\lambda - iw_2} + \ldots + \frac{b_n}{\lambda - iw_n} \qquad \ldots(1)$$

where $\operatorname{Re} a = 0$, $b \geqslant 0$, $b_k \geqslant 0$ and where the W_j are distinct real numbers. This form will be referred to very frequently in the arguments that follow. According to the proof of Theorem 5.2 in Levinson and Redheffer[3], the W_j are the roots of $1/h$ and since they are distinct, it is easy to show that the expansion (1) of $h(\lambda)$ is unique.

3. Stability of the System $X' = AX$

We need the following two lemmas.

Lemma 3.1—If f and f^* have only one root in common, then it must have zero real part.

Proof : Write $f(\lambda)$ and $f^*(\lambda)$ in the factored forms :

$$f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \ldots\ldots (\lambda - \lambda_n)$$

and

$$f^*(\lambda) = (-1)^n (\lambda + \bar{\lambda}_1)(\lambda + \bar{\lambda}_2) \ldots\ldots (\lambda + \bar{\lambda}_n).$$

Suppose that λ_1 is the only common root to f and f^*.

If $\lambda_1 = -\bar{\lambda}_j$ for some j between 2 and n, then $\lambda_j = -\bar{\lambda}_1$. Since $-\bar{\lambda}_1$ is a root to f^*, λ_j becomes another common root to f and f^* leading to a contradiction. Hence $\lambda_1 = -\bar{\lambda}_1$, or $\operatorname{Re}\lambda_1 = 0$.

Lemma 3.2—λ_1 is a common root to f and f^* if and only if it is a common root to $f + f^*$ and $f - f^*$.

Proof : It follows from the observation that, $f = \frac{1}{2}[(f + f^*) + (f - f^*)]$ and $f^* = \frac{1}{2}[(f + f^*) - (f - f^*)]$.

Now, we consider the case where A is a 2×2 complex matrix with characteristic polynomial

$$f(\lambda) = \lambda^2 + a_1 \lambda + a_2. \quad \text{Let } f^*(\lambda) = \lambda^2 - \bar{a}_1 \lambda + \bar{a}_2$$

be the paraconjugate of f.

Theorem 3.1—The system $X' = AX$ where X is a 2×2 complex matrix with no repeated zero eigenvalue, is stable if and only if one of the following two conditions hold :

1. $\operatorname{Re} a_1 > 0$ and $\operatorname{Re} a_1 \operatorname{Re}(a_1 \bar{a}_2) - (\operatorname{Im} a_2)^2 \geqslant 0$
2. $\operatorname{Re} a_1 = \operatorname{Im} a_2 = 0$ and $a_1^2 - 4a_2 \leqslant 0$, where asymptotic stability occurs only when both inequalities in 1, are strict.

PROOF : The case of asymptotic stability is settled by Theorem 3.1 Zahreddine and Elshehawey[4].

Uniform stability, or stability which is not asymptotic occurs only in each of the following cases :

Case 1—$f(\lambda) = (\lambda - i\lambda_1)(\lambda - \lambda_2)$ where $\operatorname{Re} \lambda_2 < 0$

Case 2—$f(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)$ where λ_2 is a non-zero real number. In both cases λ_1 is a possibly zero real number[2] (Theorem 9.3).

Case 1—$f(\lambda) = (\lambda - i\lambda_1)(\lambda - \lambda_2)$.

Define

$$g(\lambda) = \frac{f(\lambda)}{\lambda - i\lambda_1} = \lambda - \lambda_2.$$

Since $\operatorname{Re} \lambda_2 < 0$ $g(\lambda)$ is a Hurwitz polynomial,

$$g^*(\lambda) = -\frac{f^*(\lambda)}{\lambda - i\lambda_1} = -(\lambda + \bar{\lambda}_2).$$

Let

$$h(\lambda) = -\frac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)} = \frac{f(\lambda) + f^*(\lambda)}{f(\lambda) - f^*(\lambda)}$$

g and g^* have no roots in common[4] (Lemma 3.1)

Therefore h is a positive function[3] (Theorem 5.1).

Since $h^*(\lambda) = -h(\lambda)$, $-h^*$ is positive, and $h(\lambda)$ can be expanded uniquely as in (1), (Levinson and Redheffer[3], Theorem 5.2)

$$h(\lambda) = \frac{\lambda^2 + \operatorname{Im} a_1 \lambda + \operatorname{Re} a_2,}{\operatorname{Re} a_1 \lambda \, i \operatorname{Im} a_2} = \frac{2\lambda + \lambda_2 - \bar{\lambda}_2}{-(\lambda_2 + \bar{\lambda}_2)} \qquad \ldots(2)$$

If Re $a_1 = 0$, then $h(\lambda)$ becomes a seeond degree polynomial in λ which obviously is not true. Hence Re $a_1 \neq 0$. We execute a long division to bring $h(\lambda)$ to the form

$$h(\lambda) = \frac{i}{(\text{Re } a_1)^2} (\text{Re } a_1 \text{ Im } a_1 - \text{Im } a_2) + \frac{1}{\text{Re } a_1} \lambda$$

$$+ \frac{\text{Re } a_2 + \dfrac{\text{Im } a_2}{(\text{Re } a_1)^2} . (\text{Re } a_1 \text{ Im } a_1 - \text{Im } a_2)}{\text{Re } a_1 \lambda + i \text{ Im } a_2}$$

which when compared to (1) leads to Re $a_1 > 0$.

Since $h(\lambda)$ is a first degree polynomial in λ, the remainder of this division must be zero, implying that Re a_1 Re $(a_1 \bar{a}_2) - (\text{Im } a_2)^2 = 0$.

Conversely, assume that Re $a_1 > 0$ and Re a_1 Re $(a_1 \bar{a}_2) - (\text{Im } a_2)^2 = 0$. Take $h(\lambda)$ as in the first of the proof :

$$h(\lambda) = \frac{f(\lambda) + f^*(\lambda)}{f(\lambda) - f^*(\lambda)} = \frac{\lambda^2 + i \text{ Im } a_1 \lambda + \text{Re } a_2}{\text{Re } a_1 \lambda + i \text{ Im } a_2}$$

$$= \frac{i}{(\text{Re } a_1)^2} (\text{Re } a_1 \text{ Im } a_1 - \text{Im } a_2) + \frac{1}{(\text{Re } a_1)} \lambda \qquad \ldots(3)$$

for the remainder of this division is zero.

This shows that $f + f^*$ and $f - f^*$ have a common root, thus implying that f and f^* have a common root (Lemma 3.2) which must be unique, for otherwise $f = f^*$ leading to Re $a_1 = 0$. Therefore the common root to f and f^* has zero real part (Lemma 3.1) call it $i\lambda_1$, where λ_1 real.

Let

$$f(\lambda) = (\lambda - i\lambda_1)(\lambda - \lambda_2)$$

since Re $a_1 > 0$, by (3), $h(\lambda)$ is positive function. (Theorem 5.2).

Let

$$g(\lambda) = \frac{f(\lambda)}{\lambda - i\lambda_1} = \lambda - \lambda_2$$

then

$$g^*(\lambda) = - \frac{f^*(\lambda)}{\lambda - i\lambda_1} = -(\lambda + \bar{\lambda}_2)$$

and

$$h(\lambda) = \frac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)} .$$

Now g and g^* have no common roots, and h positive. Therefore g is a Hurwitz polynomial (Levinson and Redheffer[3], Theorem 5.1), leading to Re $\lambda_2 < 0$

Case 2— $f(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)$.

$$f(\lambda) = \lambda^2 - i(\lambda_1 + \lambda_2)\lambda - \lambda_1\lambda_2 = \lambda^2 + a_1\lambda + a_2$$

so $\lambda_1 + \lambda_2 = i\,a_1$ and $\lambda_1\lambda_2 = -a_2$.

Hence Re a_1 = Im $a_2 = 0$. it is easy to check the identity $(\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2 = (\lambda_1 - \lambda_2)^2 \geqslant 0$.

Therefore $4a_2 - a_1^2 \geqslant 0$

conversely, assume Re a_1 = Im $a_2 = 0$, and $4a_2 - a_1^2 \geqslant 0$.

Consider the quadratic equation with real coefficients $X^2 - ia_1X - a_2 = 0$. Its discriminant $-a_1^2 + 4a_2 \geqslant 0$, so it has two real roots λ_1 and λ_2 such that :

$\lambda_1 + \lambda_2 = ia_1$ and $\lambda_1\lambda_2 = -a_2$. It is easy to verify that $f(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)$.

Consider now, the system $X = AX$ where A is a 3×3 complex matrix with characteristic polynomial

$f(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$, and let $f^*(\lambda) = -\lambda^3 + \bar{a}_1\lambda^2 - \bar{a}_2\lambda + \bar{a}_3$ be the paraconjugate of f.

Define the numbers a, b and c by :

$$a = \text{Re}\, a_1 \text{ Re Re}\,(a_1\bar{a}_2 - a_3) - (\text{Im}\, a_2)^2,$$

$$b = \text{Re}\, a_1 \text{ Re}\,(a_2\bar{a}_3) - (\text{Re}\, a_3)^2, \text{ and}$$

$$c = \text{Re}\, a_1 \text{ Im}\,(\bar{a}_1 a_3) + a_3 \text{ Im}\, a_2.$$

Theorem 3.2—The system $X' = AX$ where A is a 3×3 complex matrix no repeated zero eigenvalue, is stable if and only if one of the following three conditions hold :

1. Re $a_1 > 0$, $a > 0$ and $ab - c^2 \geqslant 0$
2. Re $a_1 > 0$, $a = c = 0$ and 4 Re a_1 Re $(a_1\bar{a}_2) - 3(\text{Im}\, a_2)^2 > 0$
3. Re a_1 = Im a_2 = Re $a_3 = 0$ and there exists a number α with Re $\alpha = 0$, $f(\alpha) = 0$ and satisfying $3\alpha^2$ and $2a_1\alpha - a_1^2 + 4a_2 \geqslant 0$, where asymptotic stability occurs only when all the inequalities in 1, are strict.

PROOF : The case of asymptotic stability is settled by (Zahreddine and Elshehawey[2], Theorem 3.2). Uniform stability, or stability which is not asymptotic, occurs only in each of the following cases:

Case 1— $f(\lambda) = (\lambda - i\lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$ where Re $\lambda_2 < 0$, Re $\lambda_3 < 0$.

Case 2— $f(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - \lambda_3)$ where λ_2 is a non zero real number and Re $\lambda_3 < 0$.

Case 3— $f(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - i\lambda_3)$ where λ_2 and λ_3 are non-zero real numbers.

In all three cases, λ_1 denotes a possibly zero real number (Theorem 9.3 of Jordan and Smith[2]).

Case 1— $f(\lambda) = (\lambda - i\lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$

Define

$$g(\lambda) = \frac{f(\lambda)}{\lambda - i\lambda_1} = (\lambda - \lambda_2)(\lambda - \lambda_3)$$

g is a Hurwitz polynomial for Re $\lambda_2 < 0$ and Re $\lambda_3 < 0$

$$g^*(\lambda) = -\frac{f^*(\lambda)}{\lambda - i\lambda_1} = (\lambda + \bar{\lambda}_2)(\lambda + \bar{\lambda}_3)$$

let

$$h(\lambda) = \frac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)} = \frac{f(\lambda) + f^*(\lambda)}{f(\lambda) - f^*(\lambda)}$$

g and g^* have no roots in common (Lemma 3.1 of Zahreddine and Elshehawey[4]) and therefore h is a positive function, (Theorem 5.1 of Levinson and Redheffer[3]).

Easy to see that

$$h(\lambda) = \frac{\text{Re } a_1 \lambda^2 + i \text{ Im } a_2 \lambda + \text{Re } a_3}{\lambda^3 + i \text{ Im } a_1 \lambda^2 + \text{Re } a_2 \lambda + i \text{ Im } a_3}.$$

For any complex number $\gamma \neq 0$ it is easily checked that Re γ and Re $(1/\gamma)$ have the same sign. Hence $1/h$ is positive if h is positive. And since $h^*(\lambda) = -h(\lambda)$, then $(1/h)^* = -1/h$, hence $-(1/h)^*$ is positive. Therefore the function

$$\frac{1}{h(\lambda)} = \frac{\lambda^3 + i \text{ Im } a_1 \lambda^2 + \text{Re } a_2 \lambda + i \text{ Im } a_3}{\text{Re } a_1 \lambda^2 + i \text{ Im } a_2 \lambda + \text{Re } a_3} \quad \text{...(4)}$$

can be written as in (1), (Theorem 5.2 of Levinson and Redheffer[3])

Also,

$$\frac{1}{h(\lambda)} = \frac{\lambda^2 - i \text{ Im } (\lambda_2 + \lambda_3)\lambda + \text{Re } (\lambda_2 \lambda_3)}{-\text{Re } (\lambda_2 + \lambda_3)\lambda + i \text{ Im } (\lambda_2 \lambda_3)} \quad \text{...(5)}$$

where Re $(\lambda_2 + \lambda_3) \neq 0$.

By comparing (4) and (5) we conclude that Re $a_1 \neq 0$. In (4) we execute a long division :

$$\frac{1}{h(\lambda)} = \frac{i}{(\text{Re}a_1)^2}(\text{Re } a_1 \text{ Im } a_1 - \text{Im } a_2) + \frac{1}{\text{Re } a_1}\lambda$$

$$+ \frac{R}{\text{Re } a_1 \lambda^2 + i \text{ Im } a_2 \lambda + \text{Re } a_3},$$

where

$$R = \frac{1}{\text{Re } a_1} (a\lambda + ic)$$

a and c are defined above.

Re $a_1 > 0$ (Theorem 5.2 of Levinson and Redheffer[3]). Also Theorem 5.2 of Levinson and Redheffer[3] and the remarks that follow as illustrated by Example 5.1 in Levinson and Redheffer[3] imply that $K(\lambda) = \dfrac{\text{Re } a_1 \lambda^2 + i \text{ Im } a_2 \lambda + \text{Re } a_3}{R}$ is a positive function, and so is $-K^*(\lambda)$. Therefore $K(\lambda)$ has form (1), where again it can be shown that $a/(\text{Re } a_1)^2$ the coefficient of λ in R is non-zero.

By (5) $1/h$ reduces to a second degree polynomial in λ over a first degree one, which in turn implies that

$\dfrac{\text{Re } a_1 \lambda^2 + i \text{ Im } a_2 \lambda + \text{Re } a_3}{R}$ appears as a first degree polynomial in λ and (2) of case 1, Theorem 3.1 has re-emerged but now with different coefficients. A repetition of that argument leads to $a > 0$ and $ab - c^2 = 0$.

Conversely, assume that Re $a_1 > 0$, $a > 0$ and $ab - c^2 = 0$

$$\frac{f(\lambda) - f^*(\lambda)}{f(\lambda) + f^*(\lambda)} = \frac{\lambda^3 + i \text{ Im } a_1 \lambda^2 + \text{Re } a_2 \lambda + i \text{ Im } a_3}{\text{Re } a_1 \lambda^2 + i \text{ Im } a_2 \lambda + \text{Re } a_3} \quad \ldots(6)$$

$$= \frac{i}{(\text{Re } a_1)^2} (\text{Re } a_1 \text{ Im } a_1 - \text{Im } a_2) + \frac{1}{\text{Re } a_1} \lambda$$

$$+ \frac{R}{\text{Re } a_1 \lambda^2 + i \text{ Im } a_2 \lambda + \text{Re } a_3} \lambda$$

where R as defined before.

$$ab - c^2 = 0 \text{ implies that } \frac{R}{\text{Re } a_1 \lambda^2 + i \text{ Im } a_2 \lambda + \text{Re } a_3}$$

reduces to a non-zero constant over a first degree polynomial in λ. Therefore $f + f^*$ and $f - f^*$ have a common root which must be unique, for otherwise $(f - f^*)/(f + f^*)$ reduces to a first degree polynomial in λ, leading to $R = 0$ which is certainly not true since $a > 0$. Therefore f and f^* have only one root in common (Lemma 3.2), which must have zero real part (Lemma 3.1), call it $i\lambda_1$.

Let

$$f(\lambda) = (\lambda - i\lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3).$$

Now if

$$h(\lambda) = \frac{f(\lambda) + f^*(\lambda)}{f(\lambda) - f^*(\lambda)}, \text{ then } \frac{1}{h(\lambda)} = \frac{f(\lambda) - f^*(\lambda)}{f(\lambda) + f^*(\lambda)}$$

is a positive function since Re $a_1 > 0$ and $a > 0$. So h is also a positive function.

Let

$$g(\lambda) = \frac{f(\lambda)}{\lambda - i\lambda_1} = (\lambda - \lambda_2)(\lambda - \lambda_3)$$

then

$$g^*(\lambda) = -\frac{f^*(\lambda)}{\lambda - i\lambda_1} = (\lambda + \bar{\lambda}_2)(\lambda + \bar{\lambda}_3)$$

and

$$h(\lambda) = \frac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)}$$

g and g^* do not vanish simultaneously since $i\lambda_1$ is the only common root of f and f^*, and because $h(\lambda)$ is positive we conclude that $g(\lambda)$ is a Hurwitz polynomial (Theorem 5.1, Levinson and Redheffer[3]) therefore Re $\lambda_2 < 0$ and Re $\lambda_3 < 0$

Case 2— $f(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - \lambda_3)$

$$g(\lambda) = \frac{f(\lambda)}{(\lambda - i\lambda_1)(\lambda - i\lambda_2)} = \lambda - \lambda_3$$

is a Hurwitz polynomial since Re $\lambda < 0$

$$g^*(\lambda) = \frac{f^*(\lambda)}{(\lambda - i\lambda_1)(\lambda - i\lambda_2)} = -(\lambda + \bar{\lambda}_3).$$

Let

$$h(\lambda) = \frac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)} = \frac{f(\lambda) - f^*(\lambda)}{f(\lambda) + f^*(\lambda)}$$

h is a positive function (Lemma 3.1 Zahreddine and Elsbehawey[4] and Theorem 5.1 of Levinson and Redheffer[3]), and so is $-h^* = h$ therefore $h(\lambda)$ takes from 1), (Theorem 5.2 of Levinson and Redheffer[3])

Also h can be written :

$$h(\lambda) = \frac{\lambda^3 + i \operatorname{Im} a_1 \lambda^2 + \operatorname{Re} a_2 \lambda + i \operatorname{Im} a_3}{\operatorname{Re} a_1 \lambda^2 + i \operatorname{Im} a_2 \lambda + \operatorname{Re} a_3} = \frac{2\lambda - (\lambda_3 - \bar{\lambda}_3)}{-(\lambda_3 + \bar{\lambda}_3)} . \qquad \ldots(7)$$

It is easy to see that Re $a_1 \neq 0$. The first part of (7) leads to

$$h(\lambda) = \frac{i}{(\operatorname{Re} a_1)^2} (\operatorname{Re} a_1 \operatorname{Im} a_1 - \operatorname{Im} a_2) + \frac{1}{\operatorname{Re} a_1} \lambda$$

$$+ \frac{R}{\operatorname{Re} a_1 \lambda + i \operatorname{Im} a_2 \lambda + \operatorname{Re} a_3}.$$

By (7), $R = 0$. Since $R = \frac{1}{(\operatorname{Re} a_1)^2} (a\lambda + ic)$, then $a = c = 0$.

$\operatorname{Re} a_1 > 0$ for h is positive (Theorem 5.2 of Levinson and Redheffer[3]).

It remains to verify the inequality $4 \operatorname{Re} a_1 \operatorname{Re}(a_1 a_2) - 3 (\operatorname{Im} a_2)^2 \geqslant 0$.

Since

$$f(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - \lambda_3) = \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3$$

then

$$i\lambda_1 + i\lambda_2 + \lambda_3 = -a_1,\ i\lambda_1\lambda_3 + i\lambda_3\lambda_2 - \lambda_1\lambda_2 = a_2,\ \lambda_1\lambda_2\lambda_3 = a_3.$$

The real part of the first relation, imaginary parts of the second and third lead respectively to :

$$\operatorname{Re}\lambda_3 = -\operatorname{Re} a_1,\ (\lambda_1 + \lambda_2)\operatorname{Re}\lambda_3 = \operatorname{Im} a_2 \text{ and } \lambda_1\lambda_2 \operatorname{Im}\lambda_3 = \operatorname{Im} a_3.$$

Therefore

$$\lambda_1 + \lambda_2 = -\frac{\operatorname{Im} a_2}{\operatorname{Re} a_1} \text{ and } \lambda_1\lambda_2 = -\frac{\operatorname{Re} a_3}{\operatorname{Re} a_1}.$$

So λ_1 and λ_2 are the roots of the quadratic equation

$$\operatorname{Re} a_1 X^2 + \operatorname{Im} a_2 X - \operatorname{Re} a_3 = 0.$$

Since λ_1 and λ_2 are real $(\operatorname{Im} a_2)^2 + 4 \operatorname{Re} a_1 \operatorname{Re} a_3 \geqslant 0$, but $a = \operatorname{Re} a_1 \operatorname{Re}(a_1 \bar{a}_2 - a_3) - (\operatorname{Im} a_2)^2 = 0$ which implies $(\operatorname{Im} a_2)^2 + 4 \operatorname{Re} a_1 \operatorname{Re} a_3 = 4 \operatorname{Re} a_1 \operatorname{Re}(a_1 \bar{a}_2) - 3 (\operatorname{Im} a_2)^2$, leading therefore to the conclusion.

Conversely, suppose that $\operatorname{Re} a_1 \geqslant 0$, $a = c = 0$ and $4 \operatorname{Re} a_1 \operatorname{Re}(a_1 \bar{a}_2) - 3 (\operatorname{Im} a_2)^2 \geqslant 0$

$$\frac{f(\lambda) - f^*(\lambda)}{f(\lambda) + f^*(\lambda)} = \frac{\lambda^3 + i \operatorname{Im} a_1 \lambda^2 + \operatorname{Re} a_2 \lambda + i \operatorname{Im} a_3}{\operatorname{Re} a_1 \lambda^2 + i \operatorname{Im} a_2 \lambda + \operatorname{Re} a_3} \qquad \text{...(8)}$$

$$= \frac{1}{(\operatorname{Re} a_1)^2} (\operatorname{Re} a_1 - \operatorname{Im} a_1 - \operatorname{Im} a_2) + \frac{1}{\operatorname{Re} a_1} \lambda$$

for the remainder of this division is zero, since $a = c = 0$. Therefore $f - f^*$ and $f + f^*$ have two common roots, implying that f and f^* also have two common roots (Lemma 3.2), call them λ_1 and λ_2.

Let

$$f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

λ_3 cannot be another common root to f and f^*, for then $f = -f^*$ leading to $\operatorname{Re} a_1 = 0$

write

$$f^*(\lambda) = -(\lambda + \bar{\lambda}_1)(\lambda + \bar{\lambda}_2)(\lambda + \bar{\lambda}_3).$$

if $\lambda_1 = -\bar{\lambda}_3$ then $\lambda_3 = -\bar{\lambda}_1$, ands ince $-\bar{\lambda}_1$ is a root of f^*, λ_3 becomes a common root to f and f^* which is impossible. The same applies on λ_2.

Therefore we have only two possibilities either $\lambda_1 = -\bar{\lambda}_1$, leading to $\lambda_2 = -\bar{\lambda}_2$

or $\lambda_1 = -\bar{\lambda}_2$ which is equivalent to $\lambda_2 = -\bar{\lambda}_1$.

In both cases, we get $f^*(\lambda) = -(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda + \bar{\lambda}_3)$.

Let

$$g(\lambda) = \frac{f(\lambda)}{(\lambda - \lambda_1)(\lambda - \lambda_2)} = \lambda - \lambda_3$$

so

$$g^*(\lambda) = \frac{f^*(\lambda)}{(\lambda + \bar{\lambda}_1)(\lambda - \bar{\lambda}_2)} = \frac{f^*(\lambda)}{(\lambda - \lambda_1)(\lambda - \lambda_2)} = -(\lambda + \bar{\lambda}_3).$$

Let

$$h(\lambda) = \frac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)} = \frac{f(\lambda) - f^*(\lambda)}{f(\lambda) + f^*(\lambda)}.$$

In (8) above, $\operatorname{Re} a_1 > 0$ and $h^* = -h$ therefore $h(\lambda)$ is a positive function (Theorem 5.2 of Levinson and Redheffer[3]), and since g and g^* have no roots in common, g is a Hurwitz polynomial, (Theorem 5.1 of Levinson and Redheffer[3]). Hence $\operatorname{Re} \lambda_3 < 0$.

It remains to determine the nature of λ_1 and λ_2. When $\lambda_1 = -\bar{\lambda}_1$ and $\lambda_2 = -\lambda_2'$, then $\operatorname{Re} \lambda_1 = \operatorname{Re} \lambda_2 = 0$.

In this case the inequality $4 \operatorname{Re} a_1 \operatorname{Re}(a_1 \bar{a}_2) - 3(\operatorname{Im} a_2)^2 \geqslant 0$, is easily verified as in the previous section.

Consider now the second possibility where $\lambda_1 = -\bar{\lambda}_2$,

or $\lambda_2 = -\bar{\lambda}_1$. This leads to $\operatorname{Re} \lambda_1 = -\operatorname{Re} \lambda_2$ and

$\operatorname{Im} \lambda_1 = \operatorname{Im} \lambda_2$. Call $\beta = \operatorname{Re} \lambda_1$ and $\gamma = \operatorname{Im} \lambda_1$, so

$$\lambda_1 = \beta + i\gamma, \text{ and } \lambda_2 = -\beta + i\gamma.$$

$$f(\lambda) = [\lambda - (\beta + i\gamma)][\lambda - (-\beta + i\gamma)].(\lambda - \lambda_3)$$

$$= \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3. \text{ Then :}$$

$$2i\gamma + \lambda_3 = -a_1,\ 2i\gamma\lambda_2 - \beta^2 - \gamma^2 = a_2,\ (\beta^2 + \gamma^2)\lambda_3 = a_3.$$

Obtain λ_3 from the first relation and substitute it in the second to get : $3\gamma^2 - 2ia_1\gamma - \beta^2 - a_2 = 0$ whose real and imaginary parts produce :

$2\gamma \operatorname{Re} a_1 + \operatorname{Im} a_2 = 0$ and $\beta^2 = 3\gamma^2 + 2 \operatorname{Im} a_1 \gamma - \operatorname{Re} a_2$.

In the last relation, we substitute $\gamma = -\operatorname{Im} a_2/2\operatorname{Re} a_1$, to finally have

$$\beta^2 = \frac{3(\operatorname{Im} a_2)^2 - 4\operatorname{Re} a_1 \operatorname{Re}(a_1 \bar{a}_2)}{4(\operatorname{Re} a_1)^2}.$$

But this requires, since $\beta^2 \geqslant 0$, that $3(\operatorname{Im} a_2)^2 - 4\operatorname{Re} a_1 \operatorname{Re}(a_1 \bar{a}_2) \geqslant 0$, and by assumption, we have $3(\operatorname{Im} a_2)^2 - 4\operatorname{Re} a_1 \operatorname{Re}(a_1 \bar{a}_2) \leqslant 0$, when we compromise these two inequalities, we end up with $\beta = 0$ leading to $\operatorname{Re} \lambda_1 = 0$ and $\operatorname{Re} \lambda_2 = 0$.

Case 3— $f(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - i\lambda_3)$

$$= \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3.$$

Therefore

$$\lambda_1 + \lambda_2 + \lambda_3 = ia_1, \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = -a_2 \text{ and } \lambda_1\lambda_2\lambda_3 = -i\,a_3.$$

These relations imply $\operatorname{Re} a_1 = \operatorname{Im} a_2 = \operatorname{Re} a_3 = 0$.

The first and second relations lead to $\lambda_2 + \lambda_3 = ia_1 - \lambda_1$ and $\lambda_2\lambda_3 = -a_2 - \lambda_1(\lambda_2 + \lambda_3) = -a_2 + \lambda_1^2 - ia_1\lambda_1$.

So λ_2 and λ_3 are the roots of the quadratic equation with real coefficients $X^2 + (\lambda_1 - ia_1)X + (\lambda_1^2 - ia_1\lambda_1 - a_2) = 0$.

Since λ_2 and λ_3 are real, $(\lambda_1 - ia_1)^2 - 4(\lambda_1^2 - ia_1\lambda_1 - a_2) > 0$,

or $-3\lambda_1^2 + 2Ia_1\lambda_1 - a_1^2 + 4a_2 \geqslant 0$. Letting $\alpha = \lambda_1$, we get

$3\alpha^2 + 2a_1\alpha - a_1^2 + 4a_2 \geqslant 0$.

Conversely, suppose $\operatorname{Re} a_1 = \operatorname{Im} a_2 = \operatorname{Re} a_3 = 0$, and there exists α with $\operatorname{Re} \alpha = 0$, $f(\alpha) = 0$ and $3\alpha^2 + 2a_1\alpha - a_1^2 + 4a_2 \geqslant 0$, form the quadratic equation with real coefficients.

$X^2 - i(\alpha + a_1)X - (\alpha^2 + a_1\alpha + a_2) = 0$. The discriminant $3\alpha^2 + 2a_1\alpha - a_1^2 + ua_2 \geqslant 0$, therefore there exists two real numbers λ_2 and λ_3 such that :

$$\lambda_2 + \lambda_3 = i(\alpha + a_1) \text{ and } \lambda_2 . \lambda_3 = -(\alpha^2 + a_1\alpha + a_2)$$

also,

$$\lambda_2 . \lambda_3 = -a_2 - \alpha(\alpha + a_1) = -a_2 + i\alpha(\lambda_2 + \lambda_3).$$

Therefore,

$$-i\alpha + \lambda_2 + \lambda_3 = ia_1 \text{ and } -i\alpha\lambda_2 - i\alpha\lambda_3 + \lambda_2\lambda_3 = -a_2.$$

Since $f(\alpha) = 0$, then $\alpha^3 + a_1\alpha^2 + a_2\alpha + a_3 = 0$, and $\alpha(\alpha^2 + a_1\alpha + a_2) = -a_3$ which leads to $-\alpha\lambda_2\lambda_3 = -a_3$

or $\alpha\lambda_2\lambda_3 = a_3$. It is now straightforward to verify that

$f(\lambda) = (\lambda - \alpha)(\lambda - i\lambda_2)(\lambda - i\lambda_3)$ and the proof is complete.

Finally, we consider the system $X' = AX$ where A is a 4×4 complex matrix with characteristic polynomial,

$$f(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4$$

and let

$$f^*(\lambda) = \lambda^4 - \bar{a}_1\lambda^3 + \bar{a}_2\lambda^2 - \bar{a}_3\lambda + \bar{a}_4$$ be the paraconjugate of f.

Let a be as defined before Theorem 3.2, and define the numbers r, s and t as follows :

$$r = a.[\mathrm{Re}\, a_1 \,\mathrm{Re}\,(a_2\bar{a}_3 - a_1\bar{a}_4) - (\mathrm{Re}\, a_3)^2] - [\mathrm{Im}\, a_1 \,\mathrm{Re}\,(\bar{a_1} - a_3 a_4) + \mathrm{Re}\, a_3 \,\mathrm{Im}\, a_2]^2.$$

$$s = a.[\mathrm{Re}\, a_1 \,\mathrm{Re}\,(a_3\bar{a}_4) - (\mathrm{Im}\, a_4)^2] - [\mathrm{Re}\, a_1 \,\mathrm{Re}\,(a_1\bar{a}_4) - \mathrm{Im}\, a_2 \,\mathrm{Im}\, a_4]^2.$$

$$t = a.[\mathrm{Re}\, a_1 \,\mathrm{Im}\,(\bar{a}_2 a_4) - \mathrm{Re}\, a_3 \,\mathrm{Im}\, a_4] + [\mathrm{Re}\, a_1 \,\mathrm{Re}\,(a_1\bar{a}_4) - \mathrm{Im}\, a_2 \,\mathrm{Im} a_4].[\mathrm{Re}\, a_1 \,\mathrm{Im}\,(\bar{a}_1 a_3 - a_4) + \mathrm{Re}\, a_3 \,\mathrm{Im} a_2].$$

Theorem 3.3—The system $X' = AX$ where A is a 4×4 complex matrix with no repeated zero eigenvalue, is stable if and only if one of the following four conditions hold :

(1) $\mathrm{Re}\, a_1 > 0$, $a > 0$, $r > 0$ and $rs - t^2 \geqslant 0$.

(2) $\mathrm{Re}\, a_1 > 0$, $a > 0$, $r = t = 0$ and the inequality

$$4\,\mathrm{Re}\, a_1' \,\mathrm{Re}\,(a_1'\bar{a}_2') - 3\,(\mathrm{Im}\, a_2')^2 > 0 \text{ wher } a_1' = \gamma + a_1$$

$a_2' = \gamma^2 + a_1\gamma + a_2$, holds whenever $\mathrm{Re}\,\gamma < 0$ and $f(\gamma) = 0$.

(3) $\mathrm{Re}\, a_1 > 0, a = r = 0$, $\mathrm{Re}\, a_1 \,\mathrm{Re}\,(a_1\bar{a}_4) - \mathrm{Im}\, a_2 \,\mathrm{Im}\, a_4 = 0$, and there exists a real λ_0 with $f(i\lambda\, 0) = 0$ and satisfying

$$3\lambda_0^2 + 2\,\frac{\mathrm{Im}\, a_2}{\mathrm{Re}\, a_1}\,\lambda_0 - \frac{(\mathrm{Im}\, a_2)^2}{\mathrm{Re}\, a_1)^2} - \frac{4\,\mathrm{R}\, a_3}{\mathrm{Re}\, a_1} \leqslant 0$$

(4) Re a_1 = Im a_2 = Re a_3 = Im a_4 = 0 and there exist two numbers α and β with Re α = Re β = 0, $f(\alpha) = f(\beta) = 0$ and satisfying $3\alpha^2 + 2\alpha_1{}'\alpha - a_1' + 4a_2' > 0$

where

$$a_1' = \beta + a_1 \text{ and } a_2' = \beta^2 + a_1\beta + a_2$$

where asymptotic stability occurs only when all the inequalities in (1) are strict.

PROOF : These case of asymptotic stability is settled by Theorem 3.3 of Zahreddine and Elshehawey[4]. Non-asymptotic stability occurs only in each of the following cases.

Case 1—$f(\lambda) = (\lambda - i\lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$ where Re $\lambda_i < 0$,

$i = 2, 3, 4$.

Case 2—$f(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$ where λ_2 is a non-zero real number, Re $\lambda_3 < 0$ and Re $\lambda_4 < 0$.

Case 3—$f(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - i\lambda_3)(\lambda - \lambda_4)$, where λ_2 and λ_3 are non-zero real numbers, Re $\lambda_4 < 0$

Case 4—$f(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - i\lambda_3)(\lambda - i\lambda_4)$, where λ_2, λ_3 and λ_4 are non-zero real numbers.

In all these cases, λ_1 denotes a possibly zero real number (Theorem 9.3, Jordan and Smith[2]).

Case 1—$f(\lambda) = (\lambda - i\lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)\lambda - \lambda_4)$.

$$g^*(\lambda) = -\frac{f^*(\lambda)}{(\lambda - i\overline{\lambda_1})} = (\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4) \text{ is a Hurwitz poly-}$$

nomial since Re $\lambda_i < 0$, $i = 2, 3, 4$.

$$g(\lambda) = -\frac{f(\lambda)}{\lambda - i\lambda_1} = -(\lambda + \bar{\lambda}_2)(\lambda + \bar{\lambda}_3)(\lambda + \bar{\lambda}_4).$$

Let

$$h(\lambda) = \frac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)} = \frac{f(\lambda) + f^*(\lambda)}{f(\lambda) - f^*(\lambda)}$$

g and g^* have no common roots (Lemma 3.1 of Zahreddine and Elshehawey[4])

and consequently h is a positive function (Theorem 5.1 of Levinson and Redheffer[3]).

$$h(\lambda) = \frac{\lambda^4 + i\,\text{Im}\,a_1\lambda^3 + \text{Re}\,a_2\lambda^2 + i\,\text{Im}\,a_3\lambda + \text{Re}\,a_4}{\text{Re}\,a_1\lambda^3 + i\,\text{Im}\,a_2\lambda^2 + \text{Re}\,a_3\lambda + i\,\text{Im}\,a_4}$$

Since h is positive, it takes form (1). Therefore Re $a_1 \neq 0$. By executing a long diiv-sion, we bring $h(\lambda)$ to the form :

$$h(\lambda) = \frac{1}{\mathrm{Re}\, a_1}\lambda + \frac{i}{\mathrm{Re}\, a_1}\left(\mathrm{Im}\, a_1 - \frac{\mathrm{Im}\, a_2}{\mathrm{Re}\, a_1}\right)$$

$$+ \frac{R'}{\mathrm{Re}\, a_1 \lambda^3 + + i\, \mathrm{Im}\, a_2 \lambda^2 + \mathrm{Re}\, a_3 \lambda + i\, \mathrm{Im}\, a_4}$$

where

$$R' = \left[\left(\mathrm{Re}\, a_2 - \frac{\mathrm{Re}\, a_3}{\mathrm{Re}\, a_1}\right) + \frac{\mathrm{Im}\, a_2}{\mathrm{Re}\, a_1}\left(\mathrm{Im}\, a_1 - \frac{\mathrm{Im}\, a_2}{\mathrm{Re}\, a_1}\right)\right]\lambda^2$$

$$+ i\left[\left(\mathrm{Im}\, a_3 - \frac{\mathrm{Im}\, a_4}{\mathrm{Re}\, a_1}\right) - \frac{\mathrm{Re}\, a_3}{\mathrm{Re}\, a_1}\left(\mathrm{Im}\, a_1 - \frac{\mathrm{Im}\, a_2}{\mathrm{Re}\, a_1}\right)\right]\lambda$$

$$+ \mathrm{Re}\, a_4 + \frac{\mathrm{Im}\, a_4}{\mathrm{Re}\, a_1}\left(\mathrm{Im}\, a_1 - \frac{\mathrm{Im}\, a_2}{\mathrm{Re}\, a_1}\right).$$

Since h is a positive function, $\mathrm{Re}\, a_1 > 0$

Theorem 5.2 in Levinson and Redheffer[3] and the remarks that follow imply that

$$\frac{\mathrm{Re}\, a_1 \lambda^3 + i\, \mathrm{Im}\, a_2 \lambda^2 + \mathrm{Re}\, a_3 \lambda + i\, \mathrm{Im}\, a_4}{R'} \text{ is positive.}$$

Now,

$$h(\lambda) = \frac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)}$$

$$= \frac{(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4) + (\lambda + \bar{\lambda}_2)(\lambda + \bar{\lambda}_3)(\lambda + \bar{\lambda}_4)}{(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4) - (\lambda + \bar{\lambda}_2)(\lambda + \bar{\lambda}_3)(\lambda + \bar{\lambda}_4)}.$$

It is easy to see that $g + g^*$ is a second degree polynomial in λ, since the coefficient of λ^2 in $g + g^*$ is $-2\mathrm{Re}\,(\lambda_2 + \lambda_3 + \lambda_4)$ which is non-zero.

Therefore the positive $\dfrac{\mathrm{Re}\, a_1 \lambda^3 + i\, \mathrm{Im}\, a_2 \lambda^2 + \mathrm{Re}\, a_3 \lambda + i\, \mathrm{Im}\, a_4}{R'}$ should also reduce to a second degree polynomial in λ over a degree one. So (4) and (5) of case 1., theorem 3.2 reappear but now with different and more complicated coefficients. The process here is rather lengthy, but the argument is entirely similar and when repeated leads to $a > 0$, $r > 0$ and $rs - t^2 = 0$.

Conversely, assume that $\mathrm{Re}\, a_1 > 0$, $a > 0$, $r > 0$ and $rs - t^2 = 0$.

Take

$$\frac{f + f^*}{f - f^*} = \frac{\lambda^4 + i\, \mathrm{Im}\, a_1 \lambda^3 + \mathrm{Re}\, a_2 \lambda^2 + i\, \mathrm{Im}\, a_3 \lambda + \mathrm{Re}\, a_4}{\mathrm{Re}\, a_1 \lambda^3 + i\, \mathrm{Im}\, a_2 \lambda^2 + \mathrm{Re}\, a_3 \lambda + i\, \mathrm{Im}\, a_4}$$

$$= \frac{1}{\mathrm{Re}\, a_1}\lambda + \frac{i}{\mathrm{Re}\, a_1}\left(\mathrm{Im}\, a_1 - \frac{\mathrm{Im}\, a_2}{\mathrm{Re}\, a_1}\right)$$

$$+ \frac{R'}{\mathrm{Re}\, a_1 \lambda^3 + i\, \mathrm{Im}\, a_2 \lambda^2 + \mathrm{Re} a_3 \lambda + i\, \mathrm{Im}\, a_4}$$

where R' as defined in the first section. If we consider

$\dfrac{\operatorname{Re} a_1\lambda^3 + i \operatorname{Im} a_2 \lambda^2 + \operatorname{Re} a_3 \lambda + i \operatorname{Im} a_4}{R'}$ with the set of conditions $a > 0$, $r > 0$ and $rs - t^2 = 0$, we come to a position entirely similar to (6) in the converse of case 1, Theorem 3.2, with the same set of conditions but different and cumbersome coefficients. Another application of that argument leads to the fact that

$\operatorname{Re} a_1 \lambda^2 + i \operatorname{Im} a_2 \lambda^2 + \operatorname{Re} a_3 \lambda + i \operatorname{Im} a_4$ and R' have only one root in common. The same applies on $f + f^*$ and $f - f^*$. By Lemma 3.2 f and f^* have only one common root which must have zero real part (Lemma 3.1) call it $i\lambda_1$.

Let $f(\lambda) = (\lambda - i\lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$.

If $h(\lambda) = \dfrac{f(\lambda) + f^*(\lambda)}{f(\lambda) - f^*(\lambda)}$, then h is positive[3] (Theorem 5.2).

Let

$$g(\lambda) = \frac{f(\lambda)}{\lambda - i\lambda_1}\ (\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4),$$

$$g^*(\lambda) = -\frac{f^*(\lambda)}{\lambda - i\lambda_1} = -(\lambda + \bar{\lambda}_2)(\lambda + \bar{\lambda}_3)(\lambda + \bar{\lambda}_4)$$

so

$$h(\lambda) = \frac{f(\lambda) + f^*(\lambda)}{f(\lambda) - f^*(\lambda)} = \frac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)}$$

Since $i\lambda_1$ is the only common root to f and f^*, then g and g^* do not vanish simultaneously. Hence g is a Hurwitz polynomial[3] (Theorem 5.1). So $\operatorname{Re} \lambda_i < 0$ for $i = 2, 3, 4$.

Case 2—$f(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$

$$g(\lambda) = \frac{f(\lambda)}{(\lambda - i\lambda_1)(\lambda - i\lambda_2)} = (\lambda - \lambda_3)(\lambda - \lambda_4)$$

is a Hurwitz polynomial for $\operatorname{Re} \lambda_i < 0$: $i = 3, 4$.

$$g^*(\lambda) = \frac{f^*(\lambda)}{(\lambda - i\lambda_1)(\lambda - i\lambda_2)} = (\lambda + \bar{\lambda}_3)(\lambda + \bar{\lambda}_4).$$

Consider $h(\lambda) = \dfrac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)} = \dfrac{f(\lambda) - f^*(\lambda)}{f(\lambda) + f^*(\lambda)}$

h is positive function [Lemma 3.1[4] and Theorem 5.1[3]] and so is

$$\frac{1}{h} = \frac{\lambda^4 + i \operatorname{Im} a_1 \lambda^3 + \operatorname{Re} a_2 \lambda^2 + i \operatorname{Im} a_3 \lambda + \operatorname{Re} a_4}{\operatorname{Re} a_1 \lambda^3 + i \operatorname{Im} a_2 \lambda^2 + \operatorname{Re} a_3 \lambda + i \operatorname{Im} a_4}$$

$1/h$ takes from (1), and that leads to $\operatorname{Re} a_1 \neq 0$.

$$\frac{1}{h} = \frac{1}{\text{Re } a_1} \lambda + \frac{i}{\text{Re } a_1} (\text{Im } a_1 - \frac{\text{Im } a_2}{\text{Re } a_1})$$

$$+ \frac{R'}{\text{Re } a_1 \lambda^3 + i \text{ Im } a_2 \lambda^2 + \text{Re } a_3 \lambda + i \text{ Im } a_4}$$

where R' is already defined. By (Theorem 5.2 of Levinson and Redheffer[3]), Re $a_1 > 0$. Also,

$$\frac{1}{h} = \frac{g(\lambda) + g^*(\lambda)}{g(\lambda) - g^*(\lambda)} = \frac{(\lambda - \lambda_3)(\lambda - \lambda_4) + (\lambda + \bar{\lambda}_3)(\lambda - \bar{\lambda}_4)}{(\lambda - \lambda_3)(\lambda - \lambda_4) - (\lambda + \bar{\lambda}_3)(\lambda + \bar{\lambda}_4)}.$$

The coefficient of λ in $g - g^*$ is -2 Re $(\lambda_3 + \lambda_4)$ which is non-zero. Hence $1/h$ reduces to a second degree polynomial in λ over a first degree one which implies that Re $a_1 \lambda^3 + i$ Im $a_2 \lambda^2 +$ Re $a_3 \lambda + i$ Im a_4 must be divisible by R'.

We are now in position (7) of case 2 of Theorem 3.2 but with different coefficients. The same argument leads to $r = t = 0$, and also to $a > 0$, since

$$\frac{\text{Re } a_1 \lambda^3 + i \text{ Im } a_2 \lambda^2 + \text{Re } a_3 \lambda + i \text{ Im } a_4}{R'}$$

is a positive (Theorem 5.2 of Levinson and Redheffer[3]). Let

$$K(\lambda) = \frac{f(\lambda)}{\lambda - \lambda_4} = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - \lambda_3)$$

$$= \lambda^3 + a'_1 \lambda^2 + a'_2 \lambda + a'_3$$

where

$$a'_1 = \lambda_4 + a_1, a'_2 = \lambda_4^2 + a_1 \lambda_4 + a_2, \; a'_3 = \lambda_4^3 + a_1 \lambda_4^2 + a_2 \lambda_4 + a_3$$

$$= -\frac{a_4}{\lambda_4}.$$

But $K(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - \lambda_3)$ is precisely case 2, Theorem 3.2, and therefore the inequality 4 Re a'_1 Re $(a'_1 \bar{a}'_2) - 3(\text{Im } a'_2)^2 \geq 0$ is satisfied. The same can be proved with λ_3 replacing λ_4.

Conversely suppose Re $a_1 > 0$, $a > 0$, $r = t = 0$ and assume the above inequality holds with $a'_1 = \gamma + a_1, a'_2 = \gamma\lambda + a_1\gamma + a_2$, whenever, Re $\gamma < 0$ and $f(\gamma) = 0.$

Consider

$$\frac{f + f^*}{f - f^*} = \frac{\lambda^4 + i \text{ Im } a_1 \lambda^3 + \text{Re } a_2 \lambda^2 + i \text{ Im } a_3 \lambda + \text{Re } a_4}{\text{Re } a_1 \lambda^2 + i \text{ Im } a_2 \lambda^2 + \text{Re } a_3 \lambda + i \text{ Im } a_4}$$

(equation continued on p. 323)

$$= \frac{1}{\text{Re } a_1} \lambda + \frac{i}{\text{Re } a_1} (\text{Im } a_1 \frac{\text{Im } a_2}{\text{Re } a_1})$$

$$+ \frac{R'}{\text{Re } a_1 \lambda^3 + i \text{ Im } a_2 \lambda^2 + \text{Re } a_3 \lambda + i \text{ Im } a_4} . \qquad \text{...(9)}$$

Now

$\dfrac{\text{Re } a_1 \lambda^3 + i \text{ Im } a_2 \lambda^2 + \text{Re } a_3 \lambda + i \text{ Im } a_4}{R'}$ with $r = t = 0$ puts

us back in position (8) in the converse of case 2 of Theorem 3.2 with the same set of conditions. A similar argument implies that

$\dfrac{\text{Re } a_1 \lambda^3 + i \text{ Im } a_2 \lambda^2 + \text{Re }_3 \lambda + i \text{ Im } a_4}{R'}$ reduces to a first degree

polynomial in λ. Therefore $f + f^*$ and $f - f^*$ have two roots in common. By Lemma 3.2 f and f^* have two common roots, call them λ_1 and λ_2.

Let $f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$. λ_1 and λ_2 are the only common roots to f and f^*, for if, for instance, λ_3 is another common root to f and f^*, so $f + f^*$ and $f - f^*$ would have three roots in common, which implies that $\dfrac{f + f^*}{f - f^*}$ becomes a first degree polynomial in which is not the case.

Consider $f^*(\lambda) = (\lambda + \bar{\lambda}_1)(\lambda + \bar{\lambda}_2)(\lambda + \bar{\lambda}_3)(\lambda + \bar{\lambda}_4)$.

If $\lambda_1 = -\bar{\lambda}_3$, then $\lambda_3 = -\bar{\lambda}_1$, and since $-\bar{\lambda}_1$ is a root of $f^*(\lambda)$, λ_3 becomes a common root to f and f^* which we proved to be impossible. Similarly λ_1 cannot equal $-\bar{\lambda}_4$. The same applies on λ_2. Therefore we have only two possibilities : either $\lambda_1 = -\bar{\lambda}_1$ leading to $\lambda_2 = -\bar{\lambda}_2$, or $\lambda_1 = -\bar{\lambda}_2$ which is equivalent to $\lambda_2 = -\bar{\lambda}_1$, In both cases, we get $f^*(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda + \lambda_3)(\lambda + \bar{\lambda}_4)$

$$\text{Let } g(\lambda) = \frac{f(\lambda)}{(\lambda - \lambda_1)(\lambda - \lambda_2)} = (\lambda - \lambda_3)(\lambda - \lambda_4)$$

so

$$g^*(\lambda) = \frac{f^*(\lambda)}{(\lambda + \bar{\lambda}_1)(\lambda + \bar{\lambda}_2)} = \frac{f^*(\lambda)}{(\lambda - \lambda_1)(\lambda - \lambda_2)} = (\lambda + \bar{\lambda}_3)(\lambda + \bar{\lambda}_4).$$

Let

$$h(\lambda) = \frac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)} = \frac{f(\lambda) - f^*(\lambda)}{f(\lambda) + f^*(\lambda)}, \text{ so } \frac{1}{h(\lambda)}$$

$$= \frac{f(\lambda) + f^*(\lambda)}{f(\lambda) - f^*(\lambda)}.$$

In (9) above, since Re $a_1 > 0$, $a > 0$ and $(1/h)^* = -1/h$, $1/h$ is a positive function (Theorem 5.2 of Levinson and Reddheffer[3]) and because g and g^* have no roots in common, g is a Hurwitz polynomial[3] (Theorem 5.1). Hence Re $\lambda_3 < 0$ and and Re $\lambda_4 < 0$. It remains to determine the nature of λ_1 and λ_2.

When $\lambda_1 = -\lambda_1$ and $\lambda_2 = -\bar{\lambda}_2$ then Re $\lambda_1 = \lambda_2 = 0$.

In this case, the given inequality is easily verified as in the previous section with both λ_3 and λ_4.

Consider the second possibility where $\lambda_1 = -\bar{\lambda}_2$ or $\lambda_2 = -\bar{\lambda}_1$.

Here we have Re $\lambda_1 = -$ Re λ_2 and Im $\lambda_1 =$ Im λ_2.

since Re $\lambda_4 < 0$ and $f(\lambda_4) = 0$, the inequality,

$4 \operatorname{Re} a_1' \operatorname{Re}(a_1' \bar{a}_2') - 3(\operatorname{Im} a_2')^2 > 0$ where $a_1' = \lambda_4 + a_1$

$a_2' = \lambda_4^2 + a_1\lambda_4 + a_2$ holds.

Let

$$K(\lambda) = \frac{f(\lambda)}{\lambda - \lambda_4} = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

$$= \lambda^3 + a_1' \lambda^2 + a_2' \lambda + a_3'$$

where

$$a_3' = \lambda_4^3 + a_1 \lambda_4^2 + a_2 \lambda_4 + a_3 = -\frac{a_4}{\lambda_4}.$$

If we take up the argument towards the end of the converse of case 2 of Theorem 3.2, we can show that Re $\lambda_1 =$ Re $\lambda_2 = 0$.

Case 3— $f(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - i\lambda_3)(\lambda - \lambda_4)$

$$g(\lambda) = \frac{f^*(\lambda)}{(\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - i(\lambda_3)} = \lambda - \lambda_4$$ is a Hurwitz polynomial for Re $\lambda_4 < 0$.

$$g^*(\lambda) = -\frac{f^*(\lambda)}{(\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - i\lambda_3))} = -(\lambda + \bar{\lambda}_4).$$

The function $h(\lambda) = \dfrac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)}$ is positive (Lemma 3.1[4] and Theorem 5.1[3]) $h(\lambda)$ can also be put in the following forms :

$$h(\lambda) = \frac{\lambda^4 + i \operatorname{Im} a_1 \lambda^3 + \operatorname{Re} a_2 \lambda^2 + i \operatorname{Im} a_3 \lambda + \operatorname{Re} a_4}{\operatorname{Re} a_1 \lambda^3 + i \operatorname{Im} a_2 \lambda^2 + \operatorname{Re} a_3 \lambda + i \operatorname{Im} a_4}$$

(*equation continued on p.* 325)

$$= \frac{2\lambda + \bar{\lambda}_4 - \lambda_4}{-(\lambda_4 + \bar{\lambda}_4)} \cdot$$

Since h is positive, then (1) implies Re $a_1 \neq 0$.

Because h appears as a first degree polynomial in λ, the remainder R' of the division of $\lambda^4 + i$ Im $a_1 \lambda^3 +$ Re $a_2 \lambda^2 + i$ Im $a_3 \lambda +$ Re a_4 by Re $a_1 \lambda^3 + i$ Im $a_2 \lambda^2 +$ Re $a_3 \lambda + i$ Im a_4 mast be zero. That simply implies $a = r = 0$. and Re a_1 Re $(a_1 \bar{a}_4)$ $-$ Im a_2 Im $a_4 = 0$.

But

$$h(\lambda) = \frac{1}{\text{Re } a_1} \lambda + \frac{i}{\text{Re } a_1} (\text{Im } a_1 - \frac{\text{Im } a_2}{\text{Re } a_1})$$

shows that Re $a_1 > 0$, for h is a positive function.

Define

$$K(\lambda) = \frac{f(\lambda)}{\lambda - \lambda_4} = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - i\lambda_3)$$

$$= \lambda^3 + a'_1 \lambda^2 + a'_2 \lambda + a'_3$$

where

$$a'_1 = \lambda_4 = a_1,\ a'_2 = \lambda_4^2 + a_1\lambda_4 + a_2,\ a'_3 = \lambda_4^3 + a_1\lambda_4^2 + a_2\lambda_4 + a_3$$

$$= -\frac{a_4}{\lambda_4}$$

with $K(\lambda)$, we are in the position of case 3, Theorem 3.2. Therefore the following must be true : Re $a'_1 =$ Im a'_2 Re $a'_3 = 0$, and there exists a real λ_0 satisfying $K(i\lambda_0) = 0$, hence $f(i\lambda_0) = 0$ and $3\lambda_0^2 - 2i\, a'_2 \lambda_0 + a_1'^2 - 4a'_2 \leqslant 0$, this inequality is obtained by letting $\alpha = i\lambda_0$ in 3, Theorem 3.2.

Now Re a'_1 ·· Re $(\lambda_4 + a_1) = 0$, implies, Re $\lambda_4 = -$ Re a_1, and

$$\text{Im } a'_2 = 2 \text{ Re } \lambda_4 \text{ Im } \lambda_4 + \text{Re } a_1 \text{ Im } \lambda_4 + \text{Im } a_1 \text{ Re } \lambda_4 + \text{Im } a_2 = 0$$

when we substitute Re $\lambda_4 = -$ Re a_1 we get Im $\lambda_4 = \dfrac{\text{Im } a_2}{\text{Re } a_1} - \text{Im } a_1$.

Therefore

$$a'_1 = i\frac{\text{Im } a_2}{\text{Re } a_1}.$$

Also by substituting the values of Re λ_4 and Im λ_4 in a_2'

we get

$$a_2' = \frac{1}{(\text{Re } a_1)^2} [\text{Re } a_1 \text{ Re } (a_1 \bar{a}_2) - (\text{Im } a_2)^2]$$

and since

$$a = \text{Re } a_1 (\text{Re } a_1 - \bar{a}_2 - a_3) - (\text{Im } a_2)^2 = 0 \; a_2' = \text{Re } a_3/\text{Re } a_1.$$

Therefore λ_0 satisfies

$$3\lambda_0^2 + 2 \frac{\text{Im } a_2}{\text{Re } a_1} \lambda_0 - \frac{(\text{Im } a_2)^2}{(\text{Re } a_1)^2} - 4 \frac{\text{Re } a_3}{\text{Re } a_1} \leqslant 0.$$

Conversely, suppose Re $a_1 > 0$, $a = r = 0$ and

Re a_1 Re $(a_1 \bar{a}_4) -$ Im a_2 Im $a_4 = 0$ and assume the existence of real λ_0 with $f(i\lambda_0) = 0$ satisfying $3\lambda_0 + 2 \dfrac{\text{Im } a_2}{\text{Re } a_1} \lambda_0 - \dfrac{(\text{Im } a_2)^2}{(\text{Re } a_1)^2} - 4 \dfrac{\text{Re } a_3}{\text{Re }_1} \leqslant 0.$

Take

$$\frac{f + f^*}{f - f^*} = \frac{\lambda^4 + i \text{ Im } a_1 \lambda^3 + \text{Re } a_2 \lambda^2 + i \text{ Im } a_3 \lambda + \text{Re } a_4}{\text{Re } a_1 \lambda^3 + i \text{ Im } a_2 \lambda^2 + \text{Re } a_3 \lambda + i \text{ Im } a_4},$$

the remainder of this division is zero since $a = r = 0$ and Re a_1 Re $(a_1 \bar{a}_4) -$ Im a_2 Im $a_4 = 0$, as is easily checked.

Therefore $f + f^*$ and $f - f^*$ have three roots in common. By Lemma 3.2 f and f^* have three common roots λ_1, λ_2 and λ_3.

Let

$$f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$$

and

$$f^*(\lambda) = (\lambda + \bar{\lambda}_1)(\lambda + \bar{\lambda}_2)(\lambda + \bar{\lambda}_3)(\lambda - \bar{\lambda}_4).$$

It is obvious that λ_4 cannot be another common root to f and f^*, for then $f = f^*$ leading to Re $a_1 = 0$.

Since by assumption $i\lambda_0$ where λ_0 real, is a root of $f(\lambda)$, then it must be a common root to f and f^*. Therefore $i\lambda_0$ is different from λ_4. Suppose $\lambda_1 = i\lambda_0$.

$$f(\lambda) = (\lambda - i\lambda_0)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$$

and

$$f^*(\lambda) = (\lambda - i\lambda_0)(\lambda + \bar{\lambda}_2)(\lambda + \bar{\lambda}_3)(\lambda + \bar{\lambda}_4).$$

If $\lambda_2 = -\bar{\lambda}_4$ then $\lambda_4 = -\bar{\lambda}_2$ and since $-\bar{\lambda}_2$ is a root of $f^*(\lambda)$, then λ_4 becomes a common root to f and f^* which is impossible. So $\lambda_2 \neq -\bar{\lambda}_4$ and similarly $\lambda_3 \neq -\bar{\lambda}_4$.

It is easy to show that we have only two possibilities:

either $\lambda_2 = -\bar{\lambda}_2$ leading to $\lambda_3 = -\bar{\lambda}_3$, or $\lambda_2 = -\bar{\lambda}_3$ which is equivalent to $\lambda_3 = \bar{\lambda}_2$.

In both cases $f^*(\lambda) = (\lambda - i\lambda_0)(\lambda - \lambda_2(\lambda - \lambda_3)(\lambda + \bar{\lambda}_4)$.

Let

$$g(\lambda) = \frac{f(\lambda)}{(\lambda - i\lambda_0)(\lambda - \lambda_2)(\lambda - \lambda_3)} = (\lambda - \lambda_4)$$

then

$$g^*(\lambda) = -\frac{f^*(\lambda)}{(\lambda - i\lambda_0)(\lambda - \lambda_2)(\lambda - \lambda_3)} = -(\lambda + \bar{\lambda}_4).$$

Define

$$h(\lambda) = -\frac{f(\lambda) + f^*(\lambda)}{f(\lambda) - f^*(\lambda)} = \frac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)}.$$

Since $\mathrm{Re}\, a_1 > 0$,

$$h(\lambda) = \frac{1}{\mathrm{Re}\, a_1}\lambda + \frac{i}{\mathrm{Re}\, a_1}(\mathrm{Im}\, a_1 - \frac{\mathrm{Im}\, a_2}{\mathrm{Re}\, a_1})$$

is a positive function[3] (Theorem 5.2), and since g and g^* have no roots in common, g is a Hurwitz polynomial[3] (Theorem 5.1) implying that $\mathrm{Re}\, \lambda_4 < 0$.

Also,

$$h(\lambda) = \frac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)} = \frac{2\lambda + \bar{\lambda}_4 - \lambda_4}{-(\lambda_4 + \bar{\lambda}_4)} = -\frac{1}{\mathrm{Re}\lambda_4}\lambda^2 + i\frac{\mathrm{Im}\, \lambda_4}{\mathrm{Re}\, \lambda_4}$$

Hence,

$$\mathrm{Re}\, \lambda_4 = -\mathrm{Re}\, a_1 \text{ and } \frac{\mathrm{Im}\, \lambda_4}{\mathrm{Re}\, \lambda_4} = \frac{1}{\mathrm{Re} a_1}(\mathrm{Im}\, a_1 - \frac{\mathrm{Im}\, a_2}{\mathrm{Re}\, a_1})$$

leading to

$$\mathrm{Im}\, \lambda_4 = \frac{\mathrm{Im}\, a_2}{\mathrm{Re}\, a_1} - \mathrm{Im}\, a_1.$$

Let

$$K(\lambda) = \frac{f(\lambda)}{\lambda - \lambda_4} = (\lambda - i\lambda_0)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

$$= \lambda^3 + a_2' \lambda^2 + a_2' \lambda + a_3'$$

where

$$a'_1 = \lambda_4 + a_1,\ a'_2 = \lambda_4^2 + a_1\lambda_4 + a_2,\ a'_3 = \lambda_4^3 + a_1\lambda_4^2 + a_2\lambda_4 + a_3$$

$$= -\frac{a_4}{\lambda_4}$$

With the above values of Re λ_4 and Im λ_4, and with the use of relation Re a_1 Re $(a_1\bar{a}_4)$ − Im a_2 Im $a_4 = 0$, it is easy to show that Re a'_1 = Im a'_2 = Re a'_3 = 0. Let $\alpha = i\lambda_0$, then Re $\alpha = 0$ and $K(\alpha) = 0$. Now the given inequality takes the form :

$$3\alpha^2 + 2i\frac{\text{Im } a_2}{\text{Re } a_1}\alpha + \frac{(\text{Im } a_2)^2}{(\text{Re } a_1)^2} + 4\frac{\text{Re } a_3}{\text{Re } a_1} > 0.$$

But we know from the previous section that :

$a'_1 = i$ Im a_2/Re a_1, and a'_2 = Re a_3/Re a_1, therefore

$3\alpha^2 + 2a'_1\alpha - a'^2_1 + 4a'_2 > 0$. By 3 of Theorem 3.2, Re λ_2 = Re λ_3 = 0.

Case 4—$f(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - i\lambda_3)(\lambda - i\lambda_4)$, λ_1 is a possible zero real number, 2, 3 and 4 are non-zero real numbers.

Let

$$g(\lambda) = \frac{f(\lambda)}{\lambda - i\lambda_2} = (\lambda - i\lambda_1)(\lambda - i\lambda_3)(\lambda - i\lambda_4)$$

$$= \lambda^3 + a'_1\lambda^2 + a'_2\lambda + a'_3 \text{ where}$$

$$a'_1 = i\lambda_2 + a_1,\ a'_2 = -\lambda_2^2 + ia_1\lambda_2 + a_2 \text{ and}$$

$$a'_3 = -i\lambda_2^3 - a_1\lambda_2^2 + ia_2\lambda_2 + a_3 = -\frac{a_4}{i\lambda_2} = \frac{ia_4}{\lambda_2}$$

with $g(\lambda)$, we are in the position of case 3 of Theorem 3.2, therefore Re a'_1 = Im a'_2 = Re a'_3 = 0 which simply lead to

Re a_1 = Im a_2 = Re a_3 = Im a_4 = 0.

Theorem 3.2 also implies the existence of a number α, with Re $\alpha = 0$, $g(\alpha) = 0$, hence $f(\alpha) = 0$ and satisfying $3\alpha^2 + 2a'_1\alpha - a'^2_1\ 4a'_2 \geq 0$, obviously α coincides with either $i\lambda_1$, $i\lambda_3$ or $i\lambda_4$. If we let $\beta = i\lambda_2$, then Re $\beta = 0$, $f(\beta) = 0$ and $a'_1 = \beta + a_1$, $a'_2 = \beta^2 + a_1\beta + a_2$. Conversely, suppose Re a_1 = Im a_2 = Re a_3 = Im a_4 = 0,

and there exist α and β such that Re α = Re β = 0,

$$f(\alpha) = f(\beta) = 0 \text{ satisfying } 3\alpha^2 + 2a_1' \alpha - a_1'^2 + 4a_2' \geqslant 0$$

where

$$a_1' = \beta + a_1, a_2' = \beta^2 + a_1 \beta + a_2.$$

Let

$$K(\lambda) = \frac{f(\lambda)}{\lambda - \beta} = \lambda^3 + a_1' \lambda^2 + a_2' \lambda + a_3' . \text{ So } K(\alpha) = 0,$$

and

$$\text{Re}\, a_1' = \text{Re}\, \beta + \text{Re}\, a_1 = 0, \ \text{Im}\, a_2' = \text{Im}\, \beta^2 + a_1 \beta + a_2) = 0$$

$\text{Re}\, a_3' = \text{Re}\, (\beta^3 + a_1 \beta^2 + a_2 \beta + a_3) = 0.$ Now case 3 of Theorem 3.2 implies that all roots of $K(\lambda)$ must have zero real parts. That ends the Proof.

Finally, it is worth mentioning for the sake of applications that the arguments in all three theorems clarify the relations between the nature of the eigenvalues of the system and the coefficients of its characteristic equation.

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ON THE GRAPHOIDAL COVERING NUMBER OF A GRAPH

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A graphoidal cover of a graph $G = (V, E)$ is a collection ψ of (not necessarily open) paths in G such that (a) every path in ψ has at least two vertices (b) every vertex of G is an internal vertex of at most one path in ψ, and (c) every edge of G is in some path in ψ. The graphoidal covering number $\gamma(G)$ of G is defined to be the minimum cardinality of a graphoidal cover of G. In this paper we determine the graphoidal covering numbers of trees, complete bipartite graphs, Hamiltonian graphs and regular graphs.

1. INTRODUCTION

By a graph we mean a finite undirected graph without loops or multiple edges. We follow the notation and terminology of Harary[2]. All graphs considered in this paper are assumed to be connected graphs without isolated points.

Let $G = (V, E)$ be a graph. We denote the number of vertices in G by p and the number of edges in G by q. If $P = (u_0, u_1, u_2, \ldots, u_n)$ is a path, not necessarily open, in G then u_0 and u_n are called terminals of P and $u_1, u_2, \ldots, u_{n-1}$ are called internal vertices of P. We denote by $t(P)$ the number of internal vertices of P. The following definition of graphoidal covering number of G is given in Devadas Acharya[1].

Definition—Let $G = (V, E)$ be a graph. A graphoidal cover of G is a set ψ of (not necessarily open) paths in G satisfying the following conditions.

(1) Every path in ψ has atleast two vertices.

(2) Every vertex of G is an internal vertex of atmost one path in ψ.

(3) Every edges of G is in some path in ψ.

Let $\mathcal{G}(G)$ denote the set of all graphoidal covers of G. Then $\gamma(G) = \min_{\psi \in \mathcal{G}(G)} |\psi|$ is called the graphoidal covering number of G.

Thus $\gamma(G)$ is the minimum number of internally disjoint paths covering all the edges of G. We may further assume that every edge of G is in exactly one path of a graphoidal cover ψ so that $q = \sum_{P \in \psi} |E(P)| = |\psi| + \sum_{P \in \psi} t(P)$.

In this paper we obtain the graphoidal covering numbers of trees, complete bipartite graphs, Hamiltonian graphs and regular graphs.

MAIN RESULTS

Theorem 1—$\gamma(G) = |E(G)|$ iff $G = K_2$.

PROOF : If $G = K_2$ trivially $\gamma(G) = |E(G)| = 1$. Suppose $G \neq K_2$. Let P be a path in G such that $|E(P)| > 1$. Then $\psi = \{P\} \cup [E(G) - E(P)]$ is a graphoidal cover of G and $|\psi| < |E(G)|$ so that $\gamma(G) < |E(G)|$.

Theorem 2—Let G be a tree with n vertices of degree 1. Then $\gamma(G) = n - 1$.

PROOF : We prove the result by induction on n. When $n = 2$, G is a path and hence $\gamma(G) = 1$. Suppose the result is true for any tree with $n - 1$ vertices of degree 1. Let G be a tree with n vertices of degree 1 where $n > 1$.

Let $P = (v_0, v_1, \ldots, v_k)$ be a path in G such that $d(v_0) > 2$, $d(v_k) = 1$ and $d(v_i) = 2$ for $i = 1, 2, \ldots, k - 1$.

Then $G_1 = G - \{v_1, v_2, \ldots, v_k\}$ is a tree with $n - 1$ vertices of degree 1. Hence there exists a graphoidal cover ψ of G_1 such that $|\psi| = n - 2$. Clearly $\psi \cup \{P\}$ is a graphoidal cover for G and hence $\gamma(G) \leqslant n - 1$.

Now suppose $P_1, P_2, \ldots, P_{n-2}$ is a graphoidal cover of G. Then $p - 1 = |E(G)| = (n - 2) + \sum_{i=1}^{n-2} t(P_i)$. However $\sum_{i=1}^{n-2} t(P_i) \leqslant p - n$ which gives a contradiction. Hence $\gamma(G) = n - 1$. This completes the induction and the proof.

Corollary 1—For any tree G, $\gamma(G) \geqslant \Delta - 1$ where Δ is the maximum degree of a vertex in G.

Corollary 2—Let G be a tree with $\Delta > 2$. Let v be a vertex in G such that $d(v) = \Delta$. Then $\gamma(G) = \Delta - 1$ if and only if $d(w) = 1$ or 2 for all vertices $w \neq v$.

Theorem 3—Let ψ be a graphoidal cover of G such that every vertex v with $d(v) > 1$ is an internal vertex of a path in ψ. Then $|\psi| = \gamma$.

PROOF : Let ψ_1 be any graphoidal cover of G. Then $\sum_{P \in \psi_1} t(P) \leqslant \sum_{P \in \psi} t(P)$. Hence $q - |\psi_1| \leqslant q - |\psi|$ so that $|\psi_1| \geqslant |\psi|$. Hence $|\psi| = \gamma$.

Corollary—If there exists a graphoidal cover ψ of G such that every vertex of G is an internal vertex of a path in ψ then $\gamma(G) = q - p$.

PROOF : $q = \sum_{P \in \psi} |E(P)| = |\psi| + \sum_{P \in \psi} t(P) = |\psi| + p.$

Hence

$$\gamma(G) = |\psi| = q - P.$$

Theorem 4—Let G be a hamiltonian graph. Suppose there is a vertex v in G such that $d(v) > 3$. Then $\gamma(G) = q - p$.

PROOF : Let $C = (v = v_0, v_1, ..., v_{p-1}, v)$ be a Hamilton cycle in G. Since $d(v) > 3$, there exist vertices, say x and y, distinct from v_1 and v_{p-1} adjacent to v. Let $P = (x, v, y)$. Let S denote the set of all edges of G not covered by C and P. Then $\psi = \{C, P\} \cup S$ is a graphoidal cover for G and every vertex of G is an internal vertex of a path in ψ. Hence $\gamma(G) = q - p$.

Corollary 1—For the wheel $W_n = K_1 + C_{n-1}$ where $n \geqslant 5$, we have

$$\gamma(W_n) = q - p = n - 2.$$

Corollary 2—For the complete bipartite graph $K_{n,n}$ with $n > 3$, we have

$$\gamma(K_{n,n}) = q - p = n^2 - 2n.$$

Corollary 3—For $n > 4$, $\gamma(K_n) = q - p = \frac{1}{2} n(n - 3)$.

Theorem 5—Let G be a graph with $p > 5$. If G has a Hamilton path $P = (v_1, v_2, .., v_p)$ such that v_1 and v_p have degrees $\geqslant 3$ in G, then $\gamma(G) = q - p$.

PROOF : Similar to that of Theorem 4.

We now determine $\gamma(G)$ for regular graphs.

Any 1-regular graph is K_2 and 2-regular graph is a cycle.

Hence $\gamma(G) = 1$ if G is 1-regular or 2-regular.

Theorem 6—Let G be a k-regular graph with $k > 3$. Then $\gamma(G) = q - p$.

PROOF : It is enough to construct a collection of mutually edge disjoint and internally disjoint paths in G such that every vertex of G is an internal vertex of a path in the collection.

Let $P_1 = (u_1, u_2, ..., u_n)$ be a longest path in G so that all vertices adjacent to u_1 or u_n are already in P_1. Since $k > 3$ we can find vertices x, y, z, w in P_1 such that x, y are distinct vertices each different from u_2 and are adjacent to u_1 and z, w are distinct vertices each different from u_{n-1} and are adjacent to u_n. If x, y, z, w, u_1 and u_n are all distinct, let $P_2 = (x, u_1, y)$ and $P_3 = (z, u_n, w)$. If one of x, y coincides with u_n and one of z, w coincides with u_1, say $x = u_n$ and $z = u_1$, let $P_2' = (y, u_1, u_n, w)$.

Thus we obtain a collection $\{P_1, P_2, P_3\}$ or $\{P_1, P_2'\}$ of edge disjoint and internally disjoint paths in which $u_1, u_2, ..., u_n$ are internal vertices of one path in the collection

If these vertices exhaust all the vertices in G, the proof is complete. If not let w_1 be a vertex not lying on P_1 and let Q_1 be a longest path in G containing w_1 and internally disjoint with the paths constructed above. If the end points of Q_1 are not in P_1 we make them internal vertices of some path as before. We continue this process until all the vertices are exhausted and we obtain a required collection of paths.

We now proceed to determine the graphoidal covering number of complete bipartite graphs. $K_{1,1}$ is a path and $K_{2,2}$ is a cycle. Hence $\gamma(K_{1,1}) = \gamma(K_{2,2}) = 1$. It follows from Theorem 2 that for $n > 1$, $\gamma(K_{1,n}) = n - 1$. Also $\gamma(K_{2,3}) = 2$.

Theorem 7—Let G be the complete bipartite graph $K_{m,n}$ with $m > 2$ and $n > 2$ or $m = 2$ and $n > 3$. Then $\gamma(G) = q - p$.

PROOF : *Case i*—$m = 2$ and $n > 3$.

Let $X = \{v_1, v_2\}$ and $Y = \{w_1, w_2, \ldots, w_n\}$ be a bipartition of G. Let $P_1 = (v_1, w_1, v_2, w_2, v_1)$, $P_2 = (v_2, w_3, v_1, w_4, v_2)$ and $P_{i-2} = (v_1, w_i, v_2)$ for $i = 5, 6, \ldots, n$.

Then $\psi = \{P_1, P_2, \ldots, P_{n-2}\}$ is a graphoidal cover of G and every vertex is an internal vertex of a path in ψ.

Case ii—$m = n$ and $n > 2$.

The result follows from Theorem 4.

Case iii—$m = n + 1$ and $n > 2$.

Let $X = \{v_1, v_2, \ldots, v_{n+1}\}$ and $Y = \{w_1, w_2, \ldots, w_n\}$ be a bipartition of G.

Let $P_1 = (v_1, w_1, v_2, w_2, \ldots, v_n, w_n, v_{n+1})$,

$P_2 = (w_2, v_{n+1}, w_3)$ and $P_3 = (w_2, v_1, w_3)$.

Every vertex of G is an internal vertex of one of these paths.

Case iv—$m > n + 2$ and $n > 2$.

Let $X = \{v_1, v_2, \ldots, v_m\}$ and $Y = \{w_1, w_2, \ldots, w_n\}$ be a bipartition of G. Let $P_1 = (v_1, w_1, v_2, w_2, \ldots, v_n, w_n, v_{n+1})$, $P_2 = (w_2, v_1, w_3)$ and $Q_i = (w_2, v_{n+i}, w_3)$ where $i = 1, 2, \ldots, m - n$. Every vertex is an internal vertex of one of these paths.

Hence $\gamma(G) = q - p$.

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ON α-HAUSDORFF SUBSETS, ALMOST CLOSED MAPPINGS AND ALMOST UPPER SEMICONTINUOUS DECOMPOSITION

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The purpose of the present paper is to study some properties of α-Hausdorff subsets, almost closed mappings and almost upper semicontinuous decomposition.

1. PRELIMINARIES

Our notation is standard. No separation properties are assumed for spaces unless explicitly stated.

A subset A of a space X is 'regularly open' iff $\text{Int}\,\text{Cl}A = A$. A subset A of a space X is 'regularly closed' if $\text{Int}\ \text{Cl}A = A$ (Singal and Singal[11]).

A subset A of a space X is α-paracompact (α-nearly paracompact) iff for every open (regularly open) cover $\mathscr{U}$ of A there exists an open X-locally finite family $\mathscr{V}$ which refines $\mathscr{U}$ and covers A (Kovačević[3] and Wine[12]).

A subset A of a space X is α-paracompact (α-nearly paracompact) with respect to a subset B of X iff for every open (regularly open) cover $\mathscr{U}=\{U_i : i \in I\}$ of A there exists an open family $\mathscr{V} = \{V_j : j \in J\}$ such that :

(i) $\mathscr{V}$ refines $\mathscr{U}$,

(ii) $A \subset U\{V_j : j \in J\}$,

(iii) $\mathscr{V}$ is locally finite at each point $x \in B$.

Subsets A and B of a space X are mutually α-paracompact (mutually α-nearly paracompact) iff the subset A is α-paracompact (α-nearly paracompact) with respect to the subset B and the subset B is α-paracompact (α-nearly paracompact) with respect to the subset A (Kovačević[4]).

A subset A of a space X is α-Hausdorff iff any two points a and b, such that $a \in A$ and $b \in X \setminus A$, can be strongly separated by open sets (Kovačević).

A subset A of space X is α-regular iff for any point $a \in A$ and any open subset U of X containing a there exists an open subset V such that $a \in V \subset \text{Cl}V \subset U$ (Kovačević[6]).

A subset A of a space X is Lindelöf (nearly Lindelöf) iff every open (regularly open) cover of A has a countable subcover Kelly[1] and Kovačević.

A point p of a space X is a P-point iff $p \in \text{Int}\,(\cap \{U_n : n \in N\})$ whenever $\{U_n : n \in N\}$ is a sequence of neighbourhoods of p (Kunnen[7]).

A mapping $f : X \to Y$ is almost closed (almost open) iff for any regularly closed (regularly open) set F of $X, f(F)$ is closed (open) in Y (Singal and Singal[11]).

A decomposition $\mathscr{D}$ of a space X is upper semicontinuous (almost upper semicontinuous) iff for each D in $\mathscr{D}$ and each open (regularly open) set U containing D there exists an open set V such that $D \subset V \subset U$ and V is the union of members of $\mathscr{D}$ Kelly[1] and Kovačević.

2. On α-Hausdorff Subsets

Definition 2.1—A subset A of a space X is α-almost paracompact in respect to a subset B of X iff for every open cover $\mathscr{U} = \{U_i : i \in I\}$ of A there exists an open family $\mathscr{V} = \{V_j : j \in J\}$ such that :

(i) $\mathscr{V}$ refines $\mathscr{U}$,

(ii) $A \subset \text{Cl}\,(\cup \{V_j : j \in J\})$,

(iii) $\mathscr{V}$ is locally finite at each point $x \in B$.

Theorem 2.1—Let A be an α-Hausdorff α-almost paracompact subset with respect to each point of $X \setminus A$. Then A is closed.

Proof : Let A be any α-Hausdorff α-almost paracompact subset with respect to each point of $X \setminus A$. Let a be any point of $X \setminus A$. For each $x \in A$ there exist open sets U_x and V_x such that

$$x \in U_x,\ a \in V_x \text{ and } U_x \cap V_x = \phi.$$

Now

$$\mathscr{U} = \{U_x : x \in A\}$$

is an open covering of A. Since A is α-almost paracompact with respect to each point of $X \setminus A$, then exists an open family $\mathscr{V} = \{V_j : j \in J\}$ such that :

(i) $\mathscr{V}$ is locally finite at a,

(ii) $A \subset \text{Cl}\,(\cup \{V_j : j \in J\})$,

(iii) $\mathscr{V}$ refines $\mathscr{U}$.

Since $\mathscr{V}$ is locally finite at a there exists an open neighbourhood U of a and a finite subset J_0 of J such that

$$U \cap V_j \neq \phi \text{ for } j \in J_0 \text{ and } U \cap V_j = \phi \text{ for } j \in J \setminus J_0.$$

For each $j \in J_0$ there exists $x(j) \in A$ such that $V_j \subset U_{x(j)}$. Let

$$U_1 = U \cap (\cap \{V_{x(j)} : j \in J_0\}).$$

U_1 be an open neighbourhood of a such that

$$a \in U_1 \subset X \setminus A$$

hence, the subset $X \setminus A$ is open, i.e. the subset A is closed.

Theorem 2.2—Let A be any α-Hausdorff nearly Lindelöf subset of a space X. Let $a \in X \setminus A$ be a P-point. Then there are disjoint regularly open neighbourhoods of x and A.

Consequently, if each point of $X \setminus A$ is a P-point and A is an α-Hausdorff nearly Lindelöf subset of X, then A is closed.

PROOF: Since the subset A is α-Hausdorff, then for each point $x \in A$ there exist disjoint regularly open sets U_x and V_x such that

$$x \in U_x, a \in V_x.$$

Then

$$\mathscr{U} = \{U_x : x \in A\}$$

is a regularly open covering of A. Then there exists a countable subset A_0 of A such that

$$A \subset \cup \{U_x : x \in A_0\}.$$

Let

$$U = \cup \{U_0 : x \in A_x\} \text{ and } V = \text{Int} \cap \{V_x : x \in A_0\}.$$

Then U and V are open disjoint neighbourhoods of A and a respectively ($\alpha(U)$ and $\alpha(V) - \alpha(U) = \text{Int Cl}\, U$ — are regularly open neighbourhoods of A and a respectively).

Theorem 2.3—Let A and B be two disjoint α-Hausdorff nearly Lindelöf subsets of a space X such that each point of A and B is a P-point. Then there exist disjoint regularly open neighbourhoods of A and B respectively.

PROOF: For each point $x \in B$ there exist disjoint regularly open subsets U_x and V_x such that

$$x \in U_x \text{ and } A \subset V_x.$$

The family

$$\mathscr{U} = \{U_x : x \in B\}$$

is a regularly open covering of the subset B. Since B is nearly Lindelöf then there

exists a countable subset B_0 of B such that

$$B \subset \cup \{U_x : x \in B_0\}.$$

Let

$$U = \alpha (\cup \{U_x : x \in B_0\}) \text{ and } V = \alpha(\text{Int} (\cap \{V_x : x \in B_0 \}).$$

Then U and V are disjoint regularly open neighbourhoods of B and A respectively.

Theorem 2.4—Let A and B be any disjoint α-Hausdorff subsets of a space X such that :

(i) each point of B is a P-point,

(ii) A is nearly Lindelöf,

(iii) B is α-nearly paracompact with respect to the subset A.

Then there are disjoint regularly open neighbourhoods of A and B respectively.

PROOF : For each point $x \in A$ there exist disjoint regularly open subsets U_x and V_x such that

$x \in U_x$ and $B \subset V_x$ (Theorem 2.3 in Kovačević[4]).

The family

$$\mathcal{U} = \{ U_x : x \in A\}$$

is a regularly open covering of A, hence there exists a countable subset A_0 of A such that

$$A \subset \cup \{U_x : x \in A_0\}.$$

Let

$$U = \alpha (\cup \{U_x : x \in A_0\}) \text{ and } V = (\text{Int} (\cap \{V_x : x \in A_0\}).$$

Then U and V are disjoint regularly open neighbourhoods of A and B respectively.

3. Almost Closed Mappings

Using the similar method as in Kovačević[4] we shall prove the following theorem.

Theorem 3.1—Let X be a topological space such that each point of X is a P-point. Let $f : X \to Y$ be an almost closed mapping of the space X onto a space Y such that :

(i) for each point $y \in Y$, $f^{-1} (y)$ is an α-Hausdorff subset of X, ii for each point $y \in Y$, $f^{-1} (y)$ is nearly Lindelöf or α-nearly paracompact with respect to each subset $f^{-1} (z)$, $z \in Y$ and $z \neq y$.

Then Y is Hausdorff.

PROOF : Let y_1 and y_2 be any distinct points of Y. In any case, by hypothesis, there exist disjoint regularly open neighbourhoods U_1 and U_2 of $f^{-1}(y_1)$ and $f^{-1}(y_2)$ respectively. Since the mapping f is almost closed, then there exist open subsets of Y V_1 and V_2 containing y_1 and y_2 respectively such that

$$f^{-1}(y_1) \subset f^{-1}(V_1) \subset U_1 \,;\, f^{-1}(y_2) \subset f^{-1}(V_2) \subset U_2.$$

Hence the result.

Theorem 3.2—Let X be a topological space such that each point of X is a P-point. Let $f : X \to Y$ be an almost closed mapping of a space X onto a Lindelöf space Y such that :

(i) for each point $y \in Y$, $f^{-1}(y)$ is an α-Hausdorff subset of X,

(ii) for each point $y \in Y$ the subset $f^{-1}(y)$ is nearly Lindelöf or α-nearly paracompact with respect to each subset $f^{-1}(z)$, $z \in Y$ and $z \neq y$.

Then f is continuous.

PROOF : The proof is omitted. It is identical with the proof of Theorem 3.4 in Kovačević[4].

Theorem 3.3—If f is a closed and continuous mapping of a regular space X onto a space Y such that for each $y \in Y$ $f^{-1}(y)$ is an α-paracompact subset with respect to the subset $X \setminus f^{-1}(y)$, then Y is regular.

PROOF : Let $y \in Y$ and V be an open set containing y. Since the space X is regular and the subset $f^{-1}(y)$ is α-paracompact with respect to the subset $X \setminus f^{-1}(y)$, then, by Theorem 2.6 in Kovačević[4], there exists an open neighbourhood U of $f^{-1}(y)$ such that

$$f^{-1}(y) \subset U \subset \mathrm{Cl}U \subset f^{-1}(V).$$

Since f is closed, there exists an open set W in Y such that

$$y \in W \text{ and } f^{-1}(W) \subset U.$$

Therefore, we have

$$y \in W \subset f(U) \subset f(\mathrm{Cl}\, U) \subset V.$$

Hence

$$y \in W \subset \mathrm{Cl}\, W \subset V.$$

Hence the result.

Theorem 3.4—If f is an almost closed mapping of a regular space X onto a space Y such that for each point $y \in Y$ $f^{-1}(y)$ is an α-paracompact subset with respect to the subset $X \setminus f^{-1}(y)$, then f is closed.

PROOF : Let A be any closed subset of X and let $y \in X \setminus f(A)$. Since $f^{-1}(y) \subset X \setminus A$, there exists an open subset V such that

$$f^{-1}(y) \subset V \subset \mathrm{Cl}V \subset X \setminus A.$$

Then $\alpha(V)$ is a regularly open subset such that

$$f^{-1}(y) \subset \alpha(V) \subset \mathrm{Cl}\alpha(V) \subset X \setminus A.$$

Since f is almost closed, then there exists an open set W in Y such that

$$y \in W \text{ and } f^{-1}(y) \subset f^{-1}(W) \subset \alpha(V) \subset X \setminus A.$$

Therefore we have

$$y \in W \subset X \setminus f(A)$$

hence $Y \setminus f(A)$ is open. Then $f(A)$ is closed. Hence f is closed.

4. ALMOST UPPER SEMICONTINUOUS DECOMPOSITION

Theorem 4.1—Let X be a topological space. Let $\mathscr{D}$ be an almost upper semicontinuous decomposition of X whose members are α-Hausdorff and mutually α-nearly paracompact subsets of X. Let $\mathscr{D}$ have a quotient topology. Then $\mathscr{D}$ is a Hausdorff space.

PROOF : The projection of the space X onto the space $\mathscr{D}$ is almost closed, hence, by Theorem 3.1 in Kovačević[4]. $\mathscr{D}$ is Hausdorff.

Theorem 4.2—Let X be a topological space such that each point of X is a P-point. Let $\mathscr{D}$ be an almost upper semicontinuous decomposition of X such that :

(i) each member of $\mathscr{D}$ is an α-Hausdorff subset of X,

(ii) each member $D \in \mathscr{D}$ is nearly Lindelöf or α-nearly paracompact with respect to each-member $V \in \mathscr{D}$, $D \neq V$.

Let $\mathscr{D}$ have the quotient topology. Then $\mathscr{D}$ is Hausdorff.

PROOF : The projection of the space X onto the space $\mathscr{D}$ is almost closed, hence $\mathscr{D}$ is Hausdorff.

Theorem 4.3— Let X be a regular topological space. Let $\mathscr{D}$ be an almost upper semicontinuous decomposition of the space X such that for each $D \in \mathscr{D}$, D is an α-paracompact subset with respect to the subset $X \setminus D$. Let $\mathscr{D}$ have the quotient topology. Then (i) $\mathscr{D}$ is upper semicontinuous, (ii) $\mathscr{G}$ is regular.

PROOF : The projection of the space X onto the space $\mathscr{D}$ is almost closed, hence the projection is closed. Hence $\mathscr{D}$ is an upper semicontinuous decomposition. Since the projection is closed then $\mathscr{D}$ is regular.

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CONVEX UNIVALENT POLYNOMIALS

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In this paper, we find a necessary condition and a sufficient condition for a biquadratic polynomial to be convex univalent.

1. INTRODUCTION

Let $f(z) = z + a_2 z^2 + \ldots$ be analytic in the unit disc $E = \{z : |z| < 1\}$. Let $F(z)$ be analytic and univalent in E. Let $f(0) = F(0)$. If the image of the unit disc E under the mapping f is contained in the image of the disc E by the mapping F, then $f(z)$ is said to be subordinate to $F(z)$ and this is denoted by $f(z) \prec F(z)$.

$f(z) = z + a_2 z^2 + \ldots$ is said to be convex univalent in E if and only if $\mathrm{Re}\{1 + zf''(z)/f'(z)\} > 0$ in E.

$f(z) = z + a_2 z^2 + \ldots$ is said to be starlike univalent in E if and only if $\mathrm{Re}\{zf'(z)/f(z)\} > 0$ in E.

The Hadamard product or convolution of 2 power series $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is defined as the power series $(f * g)(z) = z \neq \sum_{n=2}^{\infty} a_n b_n z^n$.

Frank[1,2] determined the necessary condition and sufficient condition for the polynomials $\sigma z + \mu a_2 z^2$ and $\sigma z + \mu a_2 z^2 + \beta a_3 z^3$ to be convex univalent. The aim of this paper is to determine the necessary condition and sufficient condition for the biquadratic polynomial $\sigma z + \mu a_2 z^2 + \beta a_3 z^3 + \gamma a_4 z^4$ to be convex univalent and to show that

$$(1/2)\, z \prec V_3(z, f) \prec \sigma z + \mu a_2 z^2 + \alpha\beta a_3 z^3 + \gamma a_4 z^4 \prec f(z)$$

where $V_3(z, f)$ is the 3rd de la Vallee' Poussin mean of the function f given by the formula $V_3(z, f) = (3/10)\, z + (3/10)\, a_2 z^2 + (1/20)\, a_3 z^3$. In general, the nth de la Vallee' Poussin mean[3] of f is given by

$$V_n(z, f) = \binom{2n}{n}^{-1} \sum_{k=1}^{n} \binom{2n}{n+k} a_k z^k, \quad (a_1 = 1).$$

2. Sufficient Condition

Theorem 1—If $f(z) = z + bz^2 + cz^3 + dz^4$ where b, c, d are real and non-negative, then the condition

$$\left(\frac{1 + 9c - 16d}{4}\right) \geqq b \geqq \left(\frac{8c - 30d - 14cd}{1 + 5c - 32d}\right)$$

ensures that $f(z)$ is convex.

Proof : It is enough to show that $\operatorname{Re}\{1 + zf''(z)/f'(z)\} > 0$ in E. Put $z = re^{i\theta}$, $0 \leqq r < 1$, $0 \leqq \theta \leqq 2\pi$. The above condition is equivalent to

$$\begin{aligned} &64d^2 r^6 + 84cd \cos\theta\, r^5 + (48bd \cos 2\theta + 27c^2) r^4 \\ &\quad + (20d \cos 3\theta + 30bc \cos\theta) r^3 + (12c \cos 2\theta + 8b^2) r^2 \\ &\quad + 6b \cos\theta\, r + 1 \geqq 0. \end{aligned} \qquad \text{...(2.1)}$$

If $r = 0$, obviously (2.1) holds. Therefore assume that $r \neq 0$. Denote the left hand side of (2.1) by $F(\theta)$. Then

$$\begin{aligned} F'(\theta) = -r \sin\theta\, \{&84cd\, r^4 + 192bd \cos\theta\, r^3 + 30(6d - 8d \sin^2\theta + bc) r^2 \\ &+ 48c \cos\theta\, r + 6b\}. \end{aligned}$$

$$F'(\theta) = 0 \text{ gives } \sin\theta = 0$$

or

$$14cd\, r^4 - 32bd\, r^3 + (30d + 5bc) r^2 - 8c\, r + b = 0. \qquad \text{...(2.2)}$$

We shall show that

$$14cd\, r^4 - 32bd\, r^3 + (30d + 5bc) r^2 - 8c\, r + b \geqq 0. \qquad \text{...(2.3)}$$

Differentiate with respect to r and use the fact that $d \leqq 1/56$. The worst case happens when $r = 1$. This is equivalent to $14cd - 32bd + 30d + 5bc - 8c + b \geqq 0$ and this gives

$$b \geqq \frac{8c - 30d - 14cd}{1 + 5c - 32d}.$$

So when r lies in the open interval (0, 1), the inequality (2, 3) is satisfied.

It remains to show that when $F(\theta)$ takes its minimum value that (2.1) is satisfied. The value of $F(\theta)$, when $\theta = 0$, is seen to contain only positive terms for $r > 0$. Hence we have to consider (2.1) only when θ takes the value π. When $\theta = \pi$, (2.1) reduces to

$$(1 - 4br + 9cr^2 - 16dr^3)(1 - 2br + 3cr^2 - 4dr^3) \geqq 0. \qquad \text{...(2.4)}$$

The condition

$$b \geqq \frac{8c - 30d - 14cd}{1 + 5c - 32d}$$

implies that $b \geqq (3/4)\ (8c - 30d + 14cd)$. This, in turn, shows that $1 - 4br + 9cr^2 - 16dr^3 \leqq 1 - 2br + 3cr^2 - 4d^3$, $0 < r < 1$. It is enough to show that $1 - 4br + 9cr^2 - 16dr^3$ has no root in the interval $(0, 1]$. Differentiate with respect to r and use $b \geqq (3/4)\ (8c - 30d + 14cd)$. This shows that the worst case happens when $r = 1$. This condition gives $1 \geqq 4b - 9c + 16d$. Therefore $b \leqq (1 + 9c - 16d)/4$. Hence the theorem is proved.

Corollary 1—Let

$$P_4\,(z) = \frac{3 + 14\,\gamma}{4}\,z + \frac{3 + 140\,\gamma}{10}\,z^2 + \frac{1 + 90\,\gamma}{20}\,z^3 + \gamma\,z^4$$

and $0 \leqq \gamma \leqq 1/70$. Then $P_4\,(z)$ is convex in E.

PROOF : $P_4\,(z)$ is convex if $\dfrac{4\,P_4\,(z)}{3 + 14\,\gamma}$ is convex. That is

$$z + \frac{2\,(3 + 140\,\gamma)}{5\,(3 + 14\,\gamma)}\,z^2 + \frac{(1 + 90\,\gamma)}{5\,(3 + 14\,\gamma)}\,z^3 + \frac{4\,\gamma}{3 + 14\,\gamma}\,z^4$$

is convex.

The equalities

$$0 \leqq \frac{4\,\gamma}{3 + 14\,\gamma} \leqq \frac{1}{56}$$

and

$$\frac{2\,(3 + 140\,\gamma)}{5\,(3 + 14\,\gamma)} \geqq \frac{\dfrac{8 + 720\,\gamma}{5\,(3 + 14\,\gamma)} - \dfrac{120\,\gamma}{3 + 14\,\gamma} - \dfrac{56\,(1 + 90\,\gamma)\,\gamma}{5\,(3 + 14\,\gamma)^2}}{1 + \dfrac{1 + 90\,\gamma}{3 + 14\,\gamma} - \dfrac{128\,\gamma}{3 + 14\,\gamma}}$$

are equivalent to $0 \leqq \gamma \leqq (1/70)$.

Corollary 2—Let $f\,(z) = z + \sum\limits_{n=2}^{\infty} a_n\,z^n$ be convex in E. Then so is

$$(P_4 * f)\,(z) = \frac{3 + 14\,\gamma}{4}\,z + \frac{3 + 140\,\gamma}{10}\,a_2\,z^2 + \frac{1 + 90\,\gamma}{20}$$
$$\times\ a_3\,z^3 + \gamma\,a_4\,z^4.$$

This follows using the celebrated result of Ruscheweyh *et al.*[4].

Corollary 3—Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be convex in E. Then

$$(P_n * f)(z) \prec V_n(z, f) \prec f(z)$$

and

$$(1/2)\, z \prec V_3(z, f) \prec (P_4 * f)(z) \prec V_4(z, f) \prec f(z).$$

PROOF : The relation $(P_n * f)(z) \prec V_n(z, f)$ follows from Frank[2] and Ruscheweyh *et al.*[4] by taking $f = P_n$ and $\varphi = f$ and $\psi = V_n\left(z, \frac{z}{1-2}\right)$. To obtain the second reation, we note that $(P_4 * f)(z) \prec V_4(z, f)$ follows from the above relation for $n = 4$. $V_3(z, f) \prec (P_4 * f)(z)$ follows from the fact that $(P_4 * f)(z)$ is convex and thus in Theorem 6 of Frank's paper[2], we take $\varphi = P_4$, $\psi = z/(1-z)$ and $f = V(z, z/(1-z)$. That $(1/2)\, z \prec V_3(z, f)$ follows from the fact that the image of the unit disc by a convex function covers the disc of radius 1/2 in the w-plane.

3. NECESSARY CONDITIONS

Theorem 2—If for all convex $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $\sigma z + \mu a_2 z^2 + \beta a_3 z^3 + \gamma a_4 z^4$ is convex and

$$(1/2)\, z \prec V_3(f, z) \prec \sigma z + \mu a_2 z^2 + \beta a_3 z^3 + \gamma a_4 z^4 \prec f(z)$$

then

$$\sigma = (1/2) + \mu - \beta + \gamma \text{ and } \mu \leqq (1/6) + (8/3)\beta - (17/3)\gamma.$$

PROOF : With $f(z) = z/(1-z) = z + z^2 + z^3 + \ldots$ we can find the minimum value of $\sigma z + \mu a_2 z^2 + \beta a_3 z^3 + \gamma a_4 z^4$ on $|z| = 1$ whence $|\sigma z + \mu z^2 + \beta z^3 + \gamma z^4| \geqq \sigma - \mu + \beta - \gamma$. If

$(1/2)\, z \prec \sigma z + \mu z^2 + \beta z^3 + \gamma z^4$, we must have $\sigma - \mu + \beta - \gamma \leqq (1/2)$. With the same $f(z) = z/(1-z)$, if $\sigma z + \mu z^2 + \beta z^3 + \gamma z^4 \prec f(z)$ for real x, $-1 < x < 1$, we must have $\sigma x + \mu x^2 + \beta x^3 + \gamma x^4 \geqq -1/2$. Allow x to tend to -1. We get $\sigma \leqq (1/2) + \mu - \beta + \gamma$. Thus $\sigma = (1/2) + \mu - \beta + \gamma$. For real α, $-1 < \alpha < 1$, consider

$$F(z) = \frac{1+\alpha^2}{1-\alpha^2}\left[T(\alpha) - T\left(\frac{\alpha - z}{1 - \alpha z}\right)\right]$$

where $T(z) = \arc\tan z$. Then $F(z)$ is convex and maps E into the unit strip

$$\frac{1+\alpha^2}{1-\alpha^2}\{\arctan \alpha - \pi/4\} \leqq \operatorname{Re} z \leqq \frac{1+\alpha^2}{1-\alpha^2}\{\arctan \alpha + \pi/4\}.$$

The function $F(z)$ has the Taylor expansion

$$F(z) = z + \sum_{n=2}^{\infty} \frac{\sin n\theta}{n \sin \theta} \; z^n$$

where

$$\sin \theta = \frac{1 - \alpha^2}{1 + \alpha^2} \text{ and } \cos \theta = \frac{2\alpha}{1 + \alpha^2} \, .$$

If the polynomial $\{(1/2) + \mu - \beta + \gamma\} z + \mu a_2 z^2 + \beta a_3 z^3 + \gamma a_4 z^4 \prec F(z)$, we must have for real x, $-1 < x < 1$

$$\arctan \alpha - \pi/4 \leqq \{1/2 + \mu - \beta + \gamma\} \sin \theta \, x + (\mu/2) \sin 2\theta \, x^2$$
$$+ (\beta/3) \sin 3\theta \, x^3 + (\gamma/4) \sin 4\theta \, x^4.$$

But $\arctan \alpha - \pi/4 = -\theta/2$. Taking the limit as x tends to -1,

$$-\theta/2 \leqq -\{1/2 + \mu - \beta + \gamma\} \sin \theta + (\mu/2) \sin 2\theta - (\beta/3) \sin$$
$$3\theta + (\gamma/4) \sin 4\theta.$$

Taking the limit as α tends to 1 corresponds to θ tending to zero. By repeated application of L'Hospital's rule, result follows.

Theorem 3—If for all convex $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$

$$\frac{3 + 14\gamma}{4} \, z + \frac{3 + 140\gamma}{10} \, a_2 z^2 + \frac{1 + 90\gamma}{20} \, a_3 z^3 + \gamma \, a_4 z^4$$

is convex and

$$(1/2) \, z \prec V_3(f, z) \prec \frac{3 + 14\gamma}{4} \, z + \frac{3 + 140\gamma}{10} \, a_2 z^2 + \frac{1 + 90\gamma}{20} \, a_3 z^3$$
$$+ \gamma \, a_4 z^4 \prec f(z)$$

then $\gamma \leqq 1/70$.

Proof : Since $\sigma = 1/2 + \mu - \beta + \gamma$ when $\mu = 1/6 + (8/3)\beta - (17/3)\gamma$, the polynomial $f(z) = \sigma z + \mu a_2 z^2 + \beta a_3 z^3 + \gamma a_4 z^4$ becomes

$$\frac{3 + 14\gamma}{4} \, z + \frac{3 + 140\gamma}{10} \, a_2 z^2 + \frac{1 + 90\gamma}{20} \, a_3 z^3 + \gamma a_4 z^4.$$

The required result is a special case of Theorem 6 in Frank's paper[2] for $n = 4$. The proof is complete.

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MODIFIED MEANS

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(*In Memory of L. Shri L. S. Bosanquet*)

The object of the present paper is to determine a class of sequnces $\lambda = (\lambda_n)$ and absolute summability methods $| A |$ for which $| A |$ and its modified methods $| A', \lambda |$ are equivalent. The special case of Nörlund matrices is taken for special study.

1. Introduction and Notations

Let c and l denote the set of convergent and absolutely convergent sequences $x = (x_n)$. Given an infinite series $\sum_{n=0}^{\infty} a_n$ with (s_n) as the sequence of nth partial sums and an infinite matrix $A = (a_{nk})$, we write the sequence-to-sequence transformation $t = A(s)$ as

$$t_n = \sum_{k=0}^{\infty} a_{nk}\, s_k \qquad \text{...(1.1)}$$

assuming that t_n exists for each $n \geqslant 0$. We write $\Phi_n = t_n - t_{n-1}$, $t_{-1} = 0$. The series $\sum_{n=0}^{\infty} a_n$ is said to be summable (A) to the value s if $t \in c$ and $\sum_{n=0}^{\infty} \Phi_n = s$; and is said to be absolutely summable A if $\Phi \in l$ i. e. $\Sigma \mid \Phi_n \mid < \infty$. (Summation without limits means summation from 0 to ∞).

For a given sequence $\lambda = (\lambda_n)$ let τ_n be the sequence of A-transformation of $(\lambda_n a_n)$, that is,

$$\tau_n = \sum_{k=0}^{\infty} a_{nk}\, \lambda_k\, a_k. \qquad \text{...(1.2)}$$

Suppose that $\lambda_n \neq 0$ whenever $n > 0$ and let

$$B = (b_{nk}) = \left(\frac{\lambda_k\, a_{nk}}{\lambda_n} \right) \qquad \text{...(1.3)}$$

in the case $\lambda_0 = 0$, we define $a_{00} = b_{00}$. We write

$$\psi_n = \tau_n / \lambda_n = \sum_{k=0}^{\infty} b_{nk}\, a_k. \qquad \text{...(1.4)}$$

The series Σa_n is said to be summable by the 'modified A-means of weight' λ or summable (A', λ) to the value s if $\Sigma \psi_n = s$; and absolutely summable by modified A means or summable $| A', \lambda |$ if $\psi \in 1$. In the case

$$\Sigma \Phi_n = \Sigma \psi_n$$

we say that the method A and $| A |$ are identical to their modified means with weight λ.

If the matrix is lower triangular, i. e. $a_{nk} = 0 \ (k > n)$, then (1.1) can be put in the form

$$t_n = \sum_{k=0}^{n} \bar{a}_{nk} a_k \qquad \ldots(1.5)$$

where

$$\bar{a}_{nk} = \sum_{\mu=k}^{n} a_{n\mu} \qquad \ldots(1.6)$$

and

$$\Phi_n = t_n - t_{n-1} = \sum_{k=0}^{n} \overset{\Delta}{a}_{nk} a_k \qquad \ldots(1.7)$$

where

$$\hat{A} = (\hat{a}_{nk}) \text{ and } \hat{a}_{nk} = \bar{a}_{nk} - \bar{a}_{n-1,\,k}. \qquad \ldots(1.8)$$

Thus in the special case when

$$\hat{a}_{nk} = \frac{\lambda_k a_{nk}}{\lambda_n} = b_{nk} \qquad \ldots(1.9)$$

the method (A) is identical with (A', λ) mean.

Let (A_k^α) be the sequence of Cesàro coefficients determined by

$$\Sigma \ A_n^\alpha x^n = (1 - x)^{-\alpha-1} \ (| x | < 1).$$

If

$$a_{nk} = \begin{cases} \dfrac{A_{n-k}^{\alpha-1}}{A_n^\alpha} & (k \leqslant n) \\ 0 & (k > n) \end{cases} \qquad \ldots(1.10)$$

and $\alpha > -1$, $\lambda_n = n$, then (1.9) holds; for we know that

$$\frac{A^{\alpha}_{n-k}}{A^{\alpha}_{n}} - \frac{A^{\alpha}_{n-1-k}}{A^{\alpha}_{n-1}} = \frac{k}{n} A^{\alpha-1}_{n-k} \quad (\alpha > -1). \qquad \text{...(1.11)}$$

Thus the Cesàro method (C, α) is identical with its modified mean.

The method (A) is said to be 'equivalent' to its modified mean with weight λ if

$$\Sigma\, \Phi_n \text{ converges} \Leftrightarrow \Sigma\, \psi_n \text{ converges.} \qquad \text{...(1.12)}$$

Similarly $| A |$ is said to be equivalent to $| A', \lambda |$ if

$$\Phi \in 1 \Leftrightarrow \psi \in 1. \qquad \text{...(1.13)}$$

When (1.12) [respectively (1.13)] holds we write $(A) \sim (A', \lambda)$, [respectively $| A | \sim | A', \lambda |$]. We write

$$\theta\,(A) = \{\lambda \in \mathrm{R} : (A) \sim (A', \lambda)\}. \qquad \text{...(1.14)}$$

Let N be a Nörlund martix (N, p) defined by

$$\alpha_{nk} = \begin{cases} \dfrac{p_{n-k}}{P_n} & (k \leqslant n) \\ 0 & (k > n) \end{cases} \qquad \text{...(1.15)}$$

where $P_n = p_0 + p_1 + \ldots + p_n \neq 0$ for $n \geqslant 0$. Let (N', p, λ) denote the modified Nörlund mean.

When $p = (p_n)$ satisfies Kaluza condition :

$$p_n > 0,\; p_{n+1}\, p_{n-1} > p_n^2\,,\; p_{n+1} \leqslant p_n \qquad \text{...(1.16)}$$

we write $p \in \mathscr{M}$. Das[4] has shown that when $p \in \mathscr{M}$,

$$n \in \theta\,(\,|\,N\,|\,) \text{ i. e. } |\,N, p\,| \sim |\,N', p, \lambda\,| \text{ with } \lambda_n = n. \qquad \text{...(1.17)}$$

Bosanquet and Das[1] have shown that a Nörlund method is identical wth its modified Nörlund mean of weight $\lambda_n = n$ if and only if the Nörlund matrix is a Cesàro matrix.

It is easy to verify that Riesz method $(R, \mu_n, 1)$ is identical to its modified mean with weight

$$\lambda_n = \frac{\mu_n}{\mu_{n+1} - \mu_n}\,.$$

When a method is identical with its modified mean as in the case of Cesàro or Hausdorff mean [see Hardy[5], p. 247] it is an ideal situation; because the modified means are usually simpler to work with. Even when identification fails, equivalence

may sometimes hold and this serves as well. There are again Nörlund methods which are not equivalent to their modified means[1].

The knowledge of the scope of $\theta(A)$ may be helpful in dealing with the summability factor problem: Suppose that we are required to determine if $\Sigma\, \epsilon_n a_n$ is summable A (or $|A|$). If we know that $\lambda \in \theta(A)$, then instead of considering A-mean of $\Sigma\, \epsilon_n a_n$, we may consider

$$(\alpha_n) = \left(\frac{1}{\lambda_n} \Sigma\, a_{nk}\, \epsilon_k\, \lambda_k\, a_k \right).$$

Now we may choose (ϵ_n), or $(\lambda_n) \in \theta(A)$, or both, in such a way that

$$\epsilon_n \lambda_n = n \text{ or } \epsilon_n \lambda_n = 1.$$

In that case α_n reduces respectively to

$$\alpha'_n = \frac{\epsilon_n}{n} \Sigma\, a_{nk}\, k a_k$$

and

$$\alpha''_n = \epsilon_n \Sigma\, a_{nk}\, a_k$$

and obviously α'_n, α''_n are simpler to work with as the factor ϵ_n is outside the sigmas.

We need some additional definitions and notations.

The matrix $A = (a_{nk})$ is said to be normal if it is lower triangular with non-zero diagonal elements. It may be noted that the normal matrix A has a two sided normal inverse which is denoted as $A^{-1} = \left(a^{-1}_{nk} \right)$.

We write

$$\theta_{m\nu} = \sum_{k=\nu}^{m} a^{-1}_{k\nu}, \qquad \theta^*_{m\nu} = \sum_{k=\nu}^{m} | a^{-1}_{k\nu} | \qquad \ldots(1.18)$$

$$\theta_{\nu} = \sum_{k=\nu}^{\infty} a^{-1}_{k\nu}, \qquad \theta^*_{\nu} = \sum_{k=\nu}^{\infty} | a^{-1}_{k\nu} | \qquad \ldots(1.19)$$

whenever these exist.

We write

$$\Omega(n, k) = \begin{cases} \dfrac{P_{n-k}}{P_n} - \dfrac{P_{n-k-1}}{P_{n-1}}, & \text{for } k \leqslant n \\ 0, & \text{for } k > n. \end{cases} \qquad \ldots(1.20)$$

Let the sequence (c_n) be defined formally by

$$(\Sigma\, c_n\, x^n)\, (\Sigma\, p_n\, x^n) = 1 \qquad \text{...(1.21)}$$

where $p_0 \neq 0$, that is, by equations

$$\sum_{k=0}^{n} p_{n-k}\, c_k = \begin{cases} 1 & (n = 0) \\ 0 & (n > 0) \end{cases} \qquad \text{...(1.22)}$$

and let

$$c_{-1} = 0.$$

We write, for any sequence (f_n)

$$\Delta^0\, f_n = f_n,\ \Delta\, f_n = f_n - f_{n+1},\ \Delta^h f_n = \Delta\, (\Delta^{h-1} f_n)$$

$$\nabla^0 f_n = f_n,\ \nabla\, f_n = f_n - f_{n-1},\ \nabla^h f_n = \nabla\, (\nabla^{h-1} f_n)$$

for $h = 1, 2, 3, \ldots$, and define $\nabla^{-1}\, f_n = f_0 + f_1 + \ldots + f_n$. We adopt the convention that $f_n = 0$ if $n < 0$. Further we write

$$f_n^{(0)} = f_n,\ f_n^{(h)} = f_0^{(h-1)} + f_1^{(h-1)} + \ldots + f_n^{(h-1)}$$

for $h = 1, 2, \ldots$.

In Section 2 we make an attempt to discuss about the general mean $| A', \lambda |$, whereas in Section 3 and onward we confine ourselves only to modified Nörlund mean $| N', p, \lambda |$. The question of modified (A', λ) mean has however not been discussed in the paper.

2. Matrix Methods

We first obtain the following fundamental lemma.

Lemma 1—Let $A = (a_{nk})$ be a triangular matrix and let $\hat{A}$ and B be defined by (1.3) and (1.8). Suppose also that $\hat{A}^{-1}$ and B^{-1}, the inverses of $\hat{A}$ and B respectively, exist. Let

$$\hat{A}\, B^{-1} = \left(\sum_{k=r}^{n} \hat{a}_{nk}\ b_{kr}^{-1} \right) = G = (g_{n,r})$$

and

$$B\, \hat{A}^{-1} = \left(\sum_{k=r}^{n} b_{nk}\ \hat{a}_{kr}^{-1} \right) = H = (h_{n,r}).$$

(a) In order that $| A', \lambda | \Rightarrow | A |$ it is necessary and sufficient that, for all k,

$$\sum_{n=k}^{\infty} | g_{nk} | \leqslant K \qquad \text{.. (2.1)}$$

where K is an absolute positive constant not necessarily the same at each occurrence.

(b) In order that $| A | \Rightarrow | A', \lambda |$, it is necessary and sufficient that, for all k,

$$\sum_{n=k}^{\infty} | h_{n,k} | \leqslant K.$$

PROOF : We write (1.4) and (1.7) as

$$\psi = B\,(a), \;\; \Phi = \hat{A}\,(a). \qquad \text{...(2.2)}$$

Since the inverses B^{-1}, $\hat{A}^{-1}$ exist, we obtain

$$a = B^{-1}\,(\psi),\; a = \hat{A}^{-1}\,(\Phi).$$

Hence

$$\Phi = \hat{A}\,B^{-1}\,(\psi) = G\,(\psi) \qquad \text{...(2.3)}$$

and

$$\psi = B\,\hat{A}^{-1}\,(\Phi) = H\,(\Phi). \qquad \text{...(2.4)}$$

It follows from a result of Knopp and Lorentz[6] and (2.3) that

$$\psi \in 1 \Rightarrow \Phi \in 1 \text{ (i. e. } G : 1 \rightarrow 1)$$

if and only if (2.1) holds. Similarly (2.4) yields $\Phi \in 1 \Rightarrow \psi \in 1$ if and only if (2.2) holds. This completes the proof.

Now we prove

Theorem 1—Let A be a normal matrix and (λ) be a positive and non-decreasing sequence if

$$\hat{A}\,A^{-1} : 1 \rightarrow 1\,; \qquad \text{...(2.5)}$$

$$\hat{A} : 1 \rightarrow 1 \qquad \text{...(2.6)}$$

$$\sum_{n>k} \frac{\lambda_n - \lambda_k}{\lambda_n} \,|\, \Delta_k\, \hat{a}_{nk} \,| = O\left(\frac{\lambda_{k+1} - \lambda_k}{\lambda_{k+1}} \right); \qquad \text{...(2.7)}$$

and

$$\sum_{n=k}^{\infty} |\theta_{nk}| \Delta\left(\frac{1}{\lambda_n}\right) = O\left(\frac{1}{\lambda_k}\right) \qquad \ldots(2.8)$$

where (θ_{nk}) is defined in (1.18). Then

$$|A', \lambda| \Rightarrow |A|.$$

Remarks : (i) Note that the condition (2.5) means that if $x \in 1$, then $\hat{A}\ A^{-1} x \in 1$, and then this is so if and only if

$$\sum_{n=\mu}^{\infty} \left| \sum_{\nu=\mu}^{n} \hat{a}_{n\nu}\ a_{\nu\mu}^{-1} \right| \leqslant K, \text{ for } \mu = 0, 1, 2, \ldots \text{ (See Knopp and Lorentz}^{6}). \qquad \ldots(2.9)$$

Similarly (2.6) holds if and only if

$$\sum_{n=\mu}^{\infty} |\hat{a}_{n\mu}| \leqslant K \qquad \ldots(2.10)$$

for $\mu = 0, 1, 2, \ldots$.

(ii) In Lemma 3, we have discussed some simple conditions under which (2.8) holds good.

PROOF : To prove the theorem, we have to prove (by Lemma 1 (a)) that, for $\mu = 0, 1, 2, \ldots$.

$$J_\mu = \sum_{n=\mu}^{\infty} |h_{n\mu}| = \sum_{n=\mu}^{\infty} \left| \sum_{\nu=\mu}^{n} \hat{a}_{n\nu}\, b_{\nu\mu}^{-1} \right| \leqslant K.$$

Since

$$b_{\nu\mu}^{-1} = \frac{\lambda_\mu}{\lambda_\nu}\ a_{\nu\mu}^{-1}$$

$$= \frac{\lambda_\mu}{\lambda_n}\ a_{\nu\mu}^{-1} + \frac{(\lambda_n - \lambda_\nu)\,\lambda_\mu}{\lambda_\nu\,\lambda_n}\ a_{\nu\mu}^{-1}, \ \mu \leqslant \nu \leqslant n$$

we obtain

$$J_\mu \leqslant J_\mu^{(1)} + J_\mu^{(2)}$$

where, by hypothesis (2.5) and since $\lambda_n > 0$ and non-decreasing

$$J_\mu^{(1)} = \lambda_\mu \sum_{n=\mu}^{\infty} \frac{1}{\lambda_n} \left| \sum_{\nu=\mu}^{\infty} \hat{a}_{n\nu}\ a_{\nu\mu}^{-1} \right|$$

(*equation continued on p.* 354)

$$< \sum_{n=\mu}^{\infty} \left| \sum_{\nu=\mu}^{n} \hat{a}_{n\nu} \quad a_{\nu\mu}^{-1} \right| \leqslant K$$

for $\mu = 0, 1, 2, \ldots$.

$$J_{\mu}^{(2)} = \lambda_{\mu} \sum_{n=\mu}^{\infty} \frac{1}{\lambda n} \left| \sum_{\nu=\mu}^{n} \frac{\lambda n - \lambda \nu}{\lambda \nu} \hat{a}_{n\nu} \; a_{\nu\mu}^{-1} \right|$$

$$= \lambda \mu \sum_{n=\mu}^{\infty} \frac{1}{\lambda n} \left| \sum_{\nu=\mu}^{n} \Delta_{\nu} \left(\frac{\lambda \nu - \lambda \nu}{\lambda \nu} \hat{a}_{n\nu} \right) \theta_{\nu\mu} \right|$$

$$\leqslant \lambda \mu \sum_{n=\mu}^{\infty} \frac{1}{\lambda n} \left| \sum_{\nu=\mu}^{n} \lambda n \, \Delta \, (1/\lambda \nu) \, \hat{a}_{n\nu+1} \, \theta_{\nu\mu} \right|$$

$$+ \lambda_{\mu} \sum_{n=\mu}^{\infty} \frac{1}{\lambda n} \left| \sum_{\nu=\mu}^{n} \frac{\lambda n - \lambda \nu}{\lambda \nu} (\Delta_{\nu} \, \hat{a}_{n\nu}) \, \theta_{\nu\mu} \right|$$

$$= J_{\mu}^{(21)} + J_{\mu}^{(22)}, \text{ say.}$$

Now by (2.6) and (2.8) we obtain

$$J_{\mu}^{(21)} \leqslant \lambda_{\mu} \sum_{\nu=\mu}^{\infty} |\theta_{\nu\mu}| \, \Delta \left(\frac{1}{\lambda \nu} \right) \sum_{n=\nu}^{\infty} |\hat{a}_{n,\nu+1}|$$

$$\leqslant K \;\; \lambda \mu \sum_{\nu=\mu}^{\infty} |\theta_{\nu\mu}| \, \Delta \left(\frac{1}{\lambda \nu} \right)$$

$$\leqslant K, \text{ for } \mu = 0, 1, 2, \ldots$$

and by (2.7) and (2.8)

$$J_{\mu}^{(22)} < \lambda \mu \sum_{\nu=\mu}^{\infty} |\theta_{\nu\mu}| \;\; \frac{1}{\lambda \nu} \sum_{n=\nu}^{\infty} \left| \frac{\lambda n - \lambda \nu}{\lambda n} \Delta_{\nu} \, \hat{a}_{n\nu} \right|$$

$$\leqslant K \;\; \lambda \mu \sum_{\nu=\mu}^{\infty} |\theta_{\nu\mu}| \, \Delta \left(\frac{1}{\lambda \nu} \right) \leqslant K$$

for $\mu = 0, 1, 2, \ldots$.

This completes the proof of Theorem 1.

Remark : It follows from Lemma 1 (b) that a necessary and sufficient condition for $| A | \Rightarrow | A', \lambda |$ is that (2.2) should hold, i.e.

$$\sum_{n=\mu}^{\infty} \Big| \sum_{\nu=\mu}^{n} b_{n\nu} \overset{\vee}{a}{}^{-1}_{\nu\mu} \Big| \leqslant K, \mu = 0, 1, 2, \ldots$$

Clearly (2.2) holds if

$$\text{(i)} \quad \sum_{n=k}^{\infty} | b_{nk} | \leqslant K, \ k = 0, 1, 2, \ldots$$

and

$$\text{(ii)} \quad \sum_{\nu=\mu}^{\infty} | \hat{a}^{-1}_{\nu\mu} | \leqslant K, \ \mu = 0, 1, 2, \ldots.$$

However the above condition (ii) does not hold in the Cesàro case. For in that case

$$\hat{a}^{-1}_{nk} = \sum_{r=k}^{n} A^{\alpha}_{r} \quad A^{-\alpha-2}_{n-r} \ .$$

It may be remarked that so far it has not been possible for the author to determine some suitable simple conditions on a general matrix A in order that the condition (2.2) for $| A | \Rightarrow | A', \lambda |$ be satisfied so as to include both the Cesàro and the general Nörlund case. However the problem of establishing $| N', p, \lambda | \sim | N, p |$ has been tackled in a somewhat satisfactory manner in later sections.

An analysis of Conditions

We now make an analysis of conditions in Theorem 1. For this we make some preparation.

Let

$$d_{nk} = \frac{a_{n+1,k}}{a_{n,k}} \quad (0 \leqslant k \leqslant n, n = 0, 1, 2, \ldots).$$

We write $A \in \mathfrak{M}^*$ if

A is normal, $a_{nk} > 0$ for $k \leqslant n$ ...(2.11)

$d_{nk} \leqslant d_{n,k-1}$ for $0 \leqslant k \leqslant n,$ ...(2.12)

$d_{nk} \leqslant 1$ for $0 \leqslant k \leqslant n.$ ...(2.13)

The following results are known.

Lemma 2—(a) Let A satisfy conditions (2.11) and (2.12). Then its normal inverse satisfies

$$a^{-1}_{nk} \leqslant 0 \; (k < n), \; a^{-1}_{nn} = \frac{1}{a_{nn}} > 0. \qquad \text{...(2.14)}$$

(b) If A satisfies conditions (2.11) and (2 13), then

$$\sum_{k=r}^{n} a^{-1}_{nk} \geqslant 0, \; 0 \leqslant r \leqslant n, \; n = 0, 1, 2, \ldots \qquad \text{...(2.15)}$$

(c) Let $A \in \mathcal{M}^*$. Then

$$\sum_{k=0}^{n} | \, a^{-1}_{nk} \, | \leqslant 2/a_{nn} \qquad \text{.. (2.16)}$$

(d) If in addition, $\sum_{k=0}^{n} a_{nk}$ is non-decreasing as n increases, then

$$| \, a^{-1}_{nk} \, | \leqslant | \, a^{-1}_{k+1,k} \, | \; (n \geqslant k + 1). \qquad \text{...(2.17)}$$

See Peyerimhoff[8] (p. 33) for the result (2.14). For the result (2.15), see Peyerimhoff[7] (Satz[4]). The result (2.18) is a trivial deduction from (2.14) and (2.15). The result (2.17) is due to Russell[9].

Remark : The result (2.15) fails to hold under the hypotheses (2.11) and (2.12). For example, let (a_n) be a sequence of positive numbers and define

$$a_{nk} = a_n \; (0 \leqslant k \leqslant n).$$

Then the hypotheses (2.11) and (2.12) are satisfied. But if (t_n) is the A-transform of (s_n), then

$$t_n = a_n \sum_{k=0}^{n} a_k$$

so that

$$s_n = -\frac{t_{n-1}}{a_{n-1}} + \frac{t_n}{a_n}.$$

Thus

$$a^{-1}_{nk} = \begin{cases} -\dfrac{1}{a_{n-1}} & (k = n - 1) \\ \dfrac{1}{a_n} & (k = n) \\ 0 & (\text{otherwise}). \end{cases}$$

Hence for a given $n \geqslant 1$, (2.15) holds for all $k \leqslant n$ if and only if

$$a_{n-1} \geqslant a_n.$$

In the following theorem we examine the case for d_{nk} a constant.

Theorem 2—Let A be normal and let $a_{nk} \neq O$ $(k \leqslant n)$.

Then

$$d_{nk} = \alpha \ (\neq 0), \ (k \leqslant n) \qquad \ldots(2.18)$$

if and only if

$$\left.\begin{aligned} a^{-1}_{n\nu} &= O \ (n \geqslant \nu + 2) \\ a_{k+1,\, k+1} \ a^{-1}_{k+1,k} &= -\alpha. \end{aligned}\right\} \qquad \ldots(2.19)$$

On the otherhand

$$\left.\begin{aligned} a^{-1}_{nk} &= O \ (k + 2 \leqslant n \leqslant N + 1) \\ \text{and} \quad a^{-1}_{N+2,k} &\neq 0 \end{aligned}\right\} \qquad \ldots(2.20)$$

if and only if

$$\left.\begin{aligned} d_{nk} &= \alpha \ (k \leqslant n \leqslant N) \\ d_{N+1,k} &\neq d_{N+1,k+1}. \end{aligned}\right\} \qquad \ldots(2.21)$$

PROOF : Suppose that (2.19) holds. Since

$$a^{-1}_{k+1,k} \ a_{kk} + a^{-1}_{k+1,k+1} \ a_{k+1,k} = 0 \qquad \ldots(2.22)$$

we have

$$d_{kk} = \frac{a_{k+1,k}}{a_{kk}} = - \ a_{k+1,k+1} \ a^{-1}_{k+1,k} = \alpha.$$

Also we have (see Russell[9], p. 101)

$$a_{n+1,n+1} \ a^{-1}_{n+1,k} = \sum_{\nu=k+1}^{n} a_{n\nu} \ (d_{nk} - d_{n\nu}) \ a^{-1}_{\nu k} \ . \qquad \ldots(2.23)$$

Since $a^{-1}_{n+1,k} = O \ (n > k + 1)$, (2.23) reduces to :

$$a_{n,k+1} \ (d_{nk} - d_{n,k+1}) \ a^{-1}_{k+1,k} = O \ (n \geqslant k + 1) \qquad \ldots(2.24)$$

and since for $n \geqslant k + 1$, $a_{n,k+1} \neq 0$, $a^{-1}_{k+1,k} \neq 0$, we obtain from (2.24) that

$$d_{nk} = d_{n,k+1} \ (n \geqslant k+1).$$

Hence (2.18) holds.

If, on the contrary, (2.18) holds, then

$$a_{nk} = \alpha^{n-k} \, a_{kk}$$

and from the formula (2.22) and (2.23) we can reduce (2.19).

If (2.20) holds, we obtain

$$a_{N+2,N+2} \, \overset{-1}{a}_{N+2,k} = a_{N+1,N+1} \quad (d_{N+1,k} - d_{N+1,k+1}) \, \overset{-1}{a}_{k+1,k}$$

and from this we obtain (2.21). The proof that (2.21) implies (2.20) follows from (2.23).

This completes the proof.

We write $A \in \mathscr{M}^{**}$ if A satisfies conditions (2.11), (2.12) and the following :

$$a_{nk} \leqslant a_{nk+1} \ (0 \leqslant k \leqslant n-1). \qquad \ldots(2.25)$$

Now we prove :

Theorem 3—Let $A \in \mathscr{M}^{**}$. Then for all m, n and r, $m \geqslant n \geqslant r$,

(i) $\theta_{mn} \geqslant \theta_{m+1,n} \geqslant 0$; $\theta_n \geqslant 0$;

(ii) $a_{nn} \ (\theta_n^* + \theta_n) = 2$;

(iii) $0 < \sum_{k=\nu}^{r} a_{nk} \, \overset{-1}{a}_{k\nu} \leqslant a_{n\nu} \, \theta_{r\nu} \ (0 \leqslant \nu < r \leqslant n)$;

(iv) $\sum_{k=n}^{m} a_{kn} \sum_{\nu=m+1}^{\infty} | \overset{-1}{a}_{\nu n} | \leqslant 1.$

Proof : Let $r < n$. Since $a_{nk} \leqslant a_{n,k+1}$ and $\overset{-1}{a}_{k\nu} \leqslant 0 \ (k \geqslant \nu+1)$ by Lemma 2 (a), we obtain :

$$\sum_{k=\nu}^{r} a_{nk} \, \overset{-1}{a}_{k\nu} = \overset{-1}{a}_{\nu\nu} \, a_{n\nu} - | \overset{-1}{a}_{\nu+1,\nu} | \, a_{n,\nu+1} - \ldots - | \overset{-1}{a}_{r\nu} | \, a_{nr}$$

$$= a_{n\nu} \left(\overset{-1}{a}_{\nu\nu} - | \overset{-1}{a}_{\nu+1,\nu} | \, \frac{a_{n,\nu+1}}{a_{n\nu}} - \ldots - | \overset{-1}{a}_{r\nu} | \, \frac{a_{nr}}{a_{n\nu}} \right)$$

$$\leqslant a_{n\nu} \left(\overset{-1}{a}_{\nu\nu} - | \overset{-1}{a}_{\nu+1,\nu} | - \ldots - | \overset{-1}{a}_{r\nu} | \right) = a_{n\nu} \, \theta_{r,\nu}. \qquad \ldots(2.26)$$

On the otherhand

$$\sum_{k=\nu}^{r} a_{nk}\, a_{k\nu}^{-1} \geqslant \sum_{k=\nu}^{n} a_{nk}\, a_{k\nu}^{-1} = \delta_{n\nu} = \begin{cases} 1 \ (n = \nu) \\ 0 \ (n \neq \nu) \end{cases} .$$

Hence from (2.26) we obtain (iii).

Now from (iii) we obtain $\theta_{mn} > 0$ and by making $m \to \infty$, we find that θ_n exists and that $\theta_n \geqslant 0$. Also $\theta_{mn} \geqslant \theta_{m+1,n}$ is trivially true after Lemma 2(a).

Since for $n \geqslant \nu$,

$$\sum_{k=\nu}^{n} |\, a_{k\nu}^{-1} \,| + \sum_{k=\nu}^{n} a_{k\nu}^{-1} = 2a_{\nu\nu}^{-1} > 0$$

we obtain, by making $n \to \infty$,

$$\left(\theta_{\nu}^{*} + \theta_{\nu} \right) = \frac{2}{a_{\nu\nu}} .$$

Hence (ii) follows. Again

$$0 \leqslant \theta_{\nu} = \left(\sum_{k=\nu}^{m} + \sum_{k=m+1}^{\infty} \right) a_{k\nu}^{\prime -1} = \theta_{m\nu} - \sum_{k=m+1}^{\infty} |\, a_{k\nu}^{-1} \,|$$

so that

$$\sum_{k=m+1}^{\infty} |\, a_{k\nu}^{-1} \,| \leqslant \theta_{m\nu}.$$

Hence

$$\sum_{k=n}^{m} a_{kn} \sum_{r=m+1}^{\infty} |\, a_{rk}^{-1} \,| \leqslant \sum_{k=n}^{m} a_{kn}\, \theta_{mk} = 1 .$$

This completes the proof of Theorem 3.

As commented earlier, we now give the following result before we take up the study of the problem for Nörlund means.

Lemma 3—Let $A \in \mathscr{M}^{**}$ and let $\lambda_n > 0$, non-decreasing such that

$$n\,(\lambda_{n+1} - \lambda_n) = O\,(\lambda_n). \qquad \text{...(2.27)}$$

Also let

$$\sum_{\nu=\mu}^{2\mu} \theta_{\nu\mu} = O(\mu). \qquad \ldots(2.28)$$

Then (2.8) holds.

PROOF : By Theorem 3 (i)

$$0 \leqslant \theta_{2\mu,\mu} < \frac{1}{\mu} \sum_{\nu=\mu}^{2\mu} \theta_{\nu\mu}$$

and therefore (2.28) implies that

$$\theta_{2\mu,\mu} = O(1).$$

Now by Theorem 3 (i)

$$\sum_{k=\mu}^{\infty} \theta_{k\mu} \Delta\left(\frac{1}{\lambda_k}\right) = \left(\sum_{k=\mu}^{2\mu} + \sum_{k=2\mu+1}^{\infty}\right) \theta_{k\mu} \Delta\left(\frac{1}{\lambda_k}\right)$$

$$= O\left(\sum_{k=\mu}^{2\mu} \frac{\theta_{k\mu}}{k\lambda_k}\right) + \theta_{2\mu,\mu} \sum_{k=2\mu+1}^{\infty} \Delta\left(\frac{1}{\lambda_k}\right)$$

$$= O\left(\frac{1}{\lambda_\mu}\right) \frac{1}{\mu} \sum_{k=\mu}^{2\mu} \theta_{k\mu} + \theta_{2\mu,\mu} . \frac{1}{\lambda_\mu} = O\left(\frac{1}{\lambda_\mu}\right)$$

by hypotheses (2.27) and (2.28).

This completes the proof.

3. MODIFIED NÖRLUND MEANS

We shall now study conditions on λ and p such that $| N', p, \lambda | \Leftrightarrow | N, p |$.

Recall that Σa_n is said to be summable $| N', p, \lambda |$ and written as $\Sigma a_n \in | N', p, \lambda |$, if

$$\sum \frac{|\tau_n|}{\lambda_n} < \infty$$

where

$$\tau_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} \lambda_k a_k$$

We may define (N', p, λ) method to be absolutely conservative if

$$\Sigma \mid a_n \mid < \infty \Rightarrow \Sigma a_n \in \mid N', p, \lambda \mid .$$

It is easily verified that the modified Nörlund means (N', p, λ) is absolutely conservative if and only if

$$J_k = \lambda_k \sum_{n=k}^{\infty} \frac{\mid p_{n-k} \mid}{\lambda_n \mid P_n \mid} < K \qquad \ldots(3.1)$$

for $k = 0, 1, 2, \ldots$.

The following Lemma gives simpler conditions in which (3.1) holds.

Lemma 4—Let (λ_n) be positive and non-decreasing and such that

$$\sum_{n=k}^{\infty} \frac{1}{n\lambda_n} = O\left(\frac{1}{\lambda_k}\right) \qquad \ldots(3.2)$$

and let (p_n) satisfy the conditions :

(i) $P_n^* = \sum_{k=0}^{n} \mid p_k \mid = O(\mid P_n \mid)$

(ii) $(n+1)\, p_n = O(\mid P_n \mid)$.

Then (N', p, λ) is absolutely conservative.

Proof : From (3.1)

$$J_k = \left(\sum_{n=k}^{2k} + \sum_{n=2k+1}^{\infty}\right) \frac{\lambda_k \mid p_{n-k} \mid}{\lambda_n \mid P_n \mid}$$

$$< \frac{K}{\mid P_k \mid} \sum_{n=k}^{2k} \mid p_{n-k} \mid + K.\, \lambda_k \sum_{n=2k+1}^{\infty} \frac{\mid p_{n-k} \mid}{\lambda_n \mid P_{n-k} \mid}$$

$$\leqslant K + K.\, \lambda_k \sum_{n=2k+1}^{\infty} \frac{1}{(n-k)\, \lambda_{n-k}} \leqslant K$$

by hypotheses.

This completes the proof.

Remark : The hypothesis (3.2) is automatically satisfied if (λ_n) is positive and

there is a constant $\alpha > 0$ such that for $n \geqslant 1$

$$\left(\frac{\lambda_n}{n^\alpha}\right) \text{ is non-decreasing.}$$

For

$$\sum_{n=k}^{\infty} \frac{1}{n\lambda_n} = \sum_{n=k}^{\infty} \frac{n^\alpha}{n^{\alpha+1}\lambda_n} \leqslant \frac{n^k}{\lambda_k} \sum_{n=k}^{\infty} \frac{1}{n^{\alpha+1}} = O\left(\frac{1}{\lambda_k}\right).$$

We now obtain the following fundamental lemma.

Lemma 5—(a) In order that

$$| N', p, \lambda | \Rightarrow | N, p |$$

it is necessary and sufficient that, for $k \geqslant 0$,

$$J_k = \lambda_k \, | P_k | \sum_{n=k}^{\infty} \Big| \sum_{\nu=k}^{n} \frac{1}{\lambda_\nu} \, \Omega(n, \nu) \, c_{\nu-k} \Big| \leqslant K$$

where $\Omega(n, \nu)$ and (c_n) are defined by (1.20) and (1.21) respectively.

(b) In order that

$$| N, p | \Rightarrow | N', p, \lambda |$$

it is necessary and sufficient that

$$I_k = \sum_{n=k}^{\infty} \frac{1}{\lambda_n \, | P_n |} \Big| \sum_{\nu=k}^{n} \lambda_\nu \, p_{n-\nu} \Big(\sum_{\mu=\nu}^{k-1} p_\mu \, c_{\nu-\mu} - P_{k-1} \, c_{\nu-k} \Big) \Big|$$

$$\leqslant K, \; k = 1, 2, 3, \ldots.$$

Proof: When A is the (N, p) matrix, we have

$$a_{nk}^{-1} = \begin{cases} P_k \, c_{n-k} & (k \leqslant n) \\ 0 & (k > n); \end{cases}$$

$$\hat{a}_{nk} = \Omega(n, k);$$

$$b_{nk} = \frac{\lambda_k \, p_{n-k}}{\lambda_n \, P_n};$$

$$\hat{a}_{nk}^{-1} = \sum_{r=k}^{n} P_r \, (c_{n-r} - c_{n-r-1})$$

$$= - \sum_{r=0}^{k-1} P_r \, (c_{n-r} - c_{n-r-1})$$

(*equation continued on p.* 363)

$$= P_{k-1}\, c_{n-k} - \sum_{r=0}^{k-1} p_r\, c_{n-r};$$

$$b_{nk}^{-1} = \frac{\lambda_k\, P_k\, c_{n-k}}{\alpha_n}.$$

The lemma now follows from Lemma 1.

It may be noted that the condition $p_n \in \mathscr{M}$ covers the Cesàro case $p_n = A_n^{\alpha-1}$ $0 < \alpha \leqslant 1$ whereas the condition $\nabla^h p_n \in \mathscr{M}$ covers the Cesàro cases : $h < \alpha \leqslant h + 1$, $h = 0, 1, 2, \ldots$.

Now we state our main theorems.

Theorem 4—Let $\lambda_n > 0$ and increasing to ∞. Let h be a non-negative integer and let

$$\nabla^h p_n \in \mathscr{M} \qquad \ldots(3.3)$$

and

$$n^{h+1}\, \lambda_n\, \Delta^{h+1} \left(\frac{1}{\lambda_n} \right) = O\,(1). \qquad \ldots(3.4)$$

In the case $h = 0$, we further assume that (3.2) holds, i. e.

$$\sum_{n=k}^{\infty} \frac{1}{n\, \lambda_n} = O \left(\frac{1}{\lambda_k} \right).$$

Then $| N', p, \lambda | \Rightarrow | N, p |$.

Theorem 5--Let (λ_n) satisfy (3.2), hold. Let h be a non-negative integer and let $\nabla^h p_n \in \mathscr{M}$. Also let

$$n^{h+1}\, \Delta^{h+1}\, \lambda_n = O\,(\lambda_n). \qquad \ldots(3.5)$$

In the case $h = 0$, we further assume that either

$$\sum_{n=k}^{\infty} \frac{1}{nP_n} = O \left(\frac{1}{P_k} \right) \qquad \ldots(3.6)$$

or

$$\left(\frac{\lambda_n}{n} \right) \text{ is non-decreasing.} \qquad \ldots(3.7)$$

Then

$$| N, p | \Rightarrow | N', p, \lambda |.$$

Remarks : (i) It may be seen that a positive sequence λ condition (3.2) is automatically satisfied whenever (3.7) holds.

(ii) The hypotheis (3.6) is satisfied in the case $P_n = A_n^{\alpha}$, $\alpha > 0$ and is not satisfied when $P_n = \sum_{\nu=0}^{n} \frac{1}{\nu + 1}$.

Theorem 6— Let (λ_n) satisfy (3.2) and let $\lambda_n \to \infty$, $n \to \infty$. Let h be a non-negative integer such that $\nabla^h p_n \in \mathfrak{M}$ and (3.4). hold. In the case $h = 0$, we further assume that either (3.6) or (3.7) holds. Then $| N, p | \Leftrightarrow | N', p, \lambda |$.

4. Lemmas

We need the following lemmas for the proof of Theorem 4.

Lemma 6—Let h be a positive integer, $\nabla^h p_n = q_n$ and $Q_n = q_0 + q_1 + \ldots + q_n$. If $q_n > 0$ and

Q_n/q_n is non-decreasing ...(4.1)

then

P_n/p_n is non-decreasing. ...(4.2)

Proof : It is enough to prove the lemma for $h = 1$, the general result will then follow from the repeated applications of this case. Changing the notation in an obvious way and writing

$$P_n^{(1)} = P_0 + P_1 + \ldots + P_n$$

it is enough to prove that if $p_n > 0$ and P_n/p_n is non-decreasing, then $P_n^{(1)}/P_n$ is non-decreasing. The result to be proved can be put in the form that

$$P_{n+1}^{(1)} P_n - P_{n+1} P_n^{(1)} > 0. \qquad \ldots(4.3)$$

We prove (4.3) by induction on n. It is trivial that (4.3) holds when $n = 0$. Let now $n \geqslant 1$ and suppose that (4.3) holds with n replaced by $n - 1$.

We have by hypothesis

$$\frac{P_n}{p_n} \leqslant \frac{P_{n+1}}{p_{n+1}}$$

so that

$$P_n (P_{n+1} - P_n) \leqslant P_{n+1} (P_n - P_{n-1});$$

that is to say

$$P_{n+1}\, P_{n-1} \leqslant P_n^2 \,. \qquad \ldots(4.4)$$

Thus by (4.4)

$$\begin{aligned}
P^{(1)}_{n+1}\, P_n - P_{n+1}\, P^{(1)}_n &= (P^{(1)}_n + P_{n+1})\, P_n - P_{n+1}\, P^{(1)}_n \\
&= P^{(1)}_n\, P_n + P_{n+1}\,(P_n - P^{(1)}_n) \\
&= P^{(1)}_n\, P_n - P_{n+1}\, P^{(1)}_{n-1} \\
&\geqslant P^{(1)}_n\, P_n - \frac{P_n^2}{P_{n-1}}\, P^{(1)}_{n-1} \\
&= \frac{P_n}{P_{n-1}}\left(P^{(1)}_n\, P_{n-1} - P_n\, P^{(1)}_{n-1}\right) \geqslant 0
\end{aligned}$$

by the induction hypothesis.

This completes the proof of the lemma.

Lemma 7—Let h be a non-negative integer and let $\Delta^h p_n \in \mathcal{M}$. Then

(i) $c^{(h)}_0 = c_0 > 0,\ c^{(h)}_n \leqslant 0\ (n \geqslant 1)$;

(ii) $c^{(h+1)}_n \geqslant c^{(h+1)}_{n+1} \geqslant 0$ for $n > 0$;

(iii) $\displaystyle\sum_{n=0}^{\infty} c^{(h)}_n\, x^n$ is absolutely convergent for $|x| \leqslant 1$;

(iv) $\displaystyle\sum_{k=n+1}^{\infty} |c^{(h)}_k| \leqslant c^{(h+1)}_n$;

(v) $(\nabla^{h-1} p_n)\, c^{(h+1)}_n \leqslant 1$;

(vi) $(\nabla^{h-1} p_n)\, c^{(h+2)}_n \leqslant 2n+1$;

(vii) $\displaystyle 0 \leqslant \sum_{\mu=k}^{r} (\Delta^h_\mu\, p_{n-\mu})\, c^{(h)}_{\mu-k} \leqslant (\Delta^h_k\, p_{n-k})\, c^{(h+1)}_{\nu-k}\,,\ 0 \leqslant k \leqslant r \leqslant n.$

Proof : This follows from Kaluza's theorem (see Hardy[5], Theorem 22), the identity (obtained from (1.21)) :

$$\left(\sum_{n=0}^{\infty} c_n^{(h)} x^n\right)\left(\sum_{n=0}^{\infty} (\nabla^h p_n) x^n\right) = 1 \qquad \ldots(4.5)$$

and from the Lemmas 3, 4, 6 of Das[4] (see also Das[3], Lemmas 1 and 2).

Lemma 8—Let $\nabla^h p_n \in \mathfrak{M}$. Let $0 \leqslant s < h$ and $0 \leqslant r \leqslant h$. Then

(i) $\nabla^s p_n > 0$, non-decreasing and

$\nabla^s p_n \to \infty$ as $n \to \infty$;

(i) $(n + h - r + 1)\, \nabla^r p_n \leqslant (h - r + 1)\, \nabla^{r-1} p_n$;

(iii) $$\nabla^r p_n \leqslant \frac{h!\,(n + h - r)!}{(n + h)!\,(h - r)!}\, p_n$$

$$\leqslant \frac{(h + 1)!\,(n + h - r)!}{(h - r)!\,(n + h + 1)!}\, P_n = O\left(\frac{P_n}{n^{r+1}}\right);$$

(iv) $$P_n = \sum_{k=0}^{n} A_{n-k}^{h-1}\, (\Delta^{h-1} p_k) \leqslant (\nabla^{h-1} p_n)\, A_n^h;$$

and

$$P_n = \sum_{k=0}^{n} A_{n-k}^{h}\, \nabla^h p_k > \nabla^h p_n.\ A_n^{h+1};$$

(v) $$\frac{\nabla^h p_n}{\nabla^{h-1} p_n} = O\left(\frac{p_n}{P_n}\right);$$

(vi) The method (N, p) is regular and absolutely regular.

Proof : Since

$$\nabla^r p_n = \sum_{\nu=0}^{n} A_{n-\nu}^{h-r-1}\, \nabla^h p_\nu \qquad \ldots(4.6)$$

and since $\nabla^h p_n > 0$, $A_\nu^{h-r-1} > 0$, the result (i) follows. To prove (ii), we proceed as in Das[3] (Lemma 3). It follows from (4.6) that

$$(h - r + 1)\, \nabla^{r-1} p_n - (n + h - r + 1)\, \nabla^r p_n$$

$$= \sum_{\nu=0}^{n} \{(h - r + 1)\, A_{n-\nu}^{h-r} - (n + h - r + 1)\, A_{n-\nu}^{h-r-1}\}\, \nabla^h p_\nu. \qquad \ldots(4.7)$$

But

$$\sum_{\nu=0}^{n} \{(h-r+1)\, A_{n-\nu}^{h-r} - (n+h-r+1)\, A_{n-\nu}^{h-r-1}\}$$

$$= (h-r+1)\, A_{n}^{h-r+1} - (n+h-r+1)\, A_{n}^{h-r} = 0 \qquad \ldots(4.8)$$

as

$$\frac{A_n^{h-r+1}}{A_n^{h-r}} = \frac{n+h-r+1}{h-r+1}$$

which is an increasing function of n. Hence it follows that there exists $\nu \leqslant \nu_0\,(n)$ such that the expression bracketed in (4.7) is non-negative for $\nu \leqslant \nu_0$ and negative for $\nu > \nu_0$. Hence the right of (4.7) is greater than or equal to

$$\sum_{\nu=0}^{n} \{(h-r+1)\, A_{n-\nu}^{h-r} - (n+h-r+1)\, A_{n-\nu}^{h-r-1}\} \, \nabla^h p_{\nu_0}$$

which is 0 by another application of (4.8). Now (ii) follows. (iii) follows by repeated application of (ii). (iv) is obvious. (vi) follows from (iii) and (iv).

Since $p_n > 0$, the necessary and sufficient condition for (N, p) to be regular is that

$$p_n = o\,(P_n) \qquad \ldots(4.9)$$

and this holds because of (iii). To prove the absolute regularity, in view of (4.9) it is enough to show that, uniformly in $\nu \geqslant 0$,

$$\sum_{n=\nu}^{\infty} |\, \Omega\,(n, \nu)\,| = O\,(1). \qquad \ldots(4.10)$$

Since

$$\Omega\,(n, \nu) = \frac{P_{n-\nu}}{P_n} - \frac{P_{n-\nu-1}}{P_{n-1}}$$

$$= \frac{P_n\, p_{n-\nu} - P_{n-\nu}\, p_n}{P_n\, P_{n-1}}$$

$$= \frac{p_n\, p_{n-\nu} \left(\dfrac{P_n}{p_n} - \dfrac{P_{n-\nu}}{p_{n-\nu}} \right)}{P_n\, P_{n-1}}$$

it follows that, as $p_n \geqslant 0$.

$$\Omega\,(n, \nu) \geqslant 0$$

if and only if P_n/p_n is non-decreasing. But since $\nabla^h p_n = q_n$ is non-increasing, it follows that Q_n/q_n is non-decreasing. Hence, by Lemma 6, it follows that P_n/p_n is non-decreasing. Hence $\Omega(n, \nu) \geqslant 0$. So

$$\sum_{n=\nu}^{\infty} |\,\Omega(n, \nu)\,| = \sum_{n=\nu}^{\infty} \Omega(n, \nu) = \lim_{m\to\infty} \frac{P_{m-\nu}}{P_m} = 1$$

by use of (4.9).

This completes the proof.

Lemma 9—(*a*) Let h be a non-negative integer such that $\nabla^h p_n \in \mathscr{M}$. If $h = 0$ we further assume that (3.6) holds. Then

$$P_k \sum_{\nu=k}^{\infty} \frac{c_{\nu-k}^{(h+1)}}{(\nu+1)^{h+1}} = O(1).$$

(b) Let $\nabla^h p_n \in \mathscr{M}$ and let (λ_n) satisfy (3.2). Then

$$\lambda_k\, P_k \sum_{\nu=k}^{\infty} \frac{c_{\nu-k}^{(h+1)}}{(\nu+1)^{h+1}\lambda_\nu} = O(1).$$

Proof : Let $h \geqslant 1$. Since $c_n^{(h+1)}$ is non-increasing we obtain

$$\sum_{\nu=k}^{\infty} \frac{c_{\nu-k}^{(h+1)}}{(\nu+1)^{h+1}} = \left(\sum_{\nu=k}^{2k} + \sum_{\nu=2k+1}^{\infty}\right) \frac{c_{\nu-k}^{(h+1)}}{(\nu+1)^{h+1}}$$

$$\leqslant \frac{c_k^{(h+2)}}{(k+1)^{h+1}} + c_k^{(h+1)} \sum_{\nu=2k+1}^{\infty} \Big) \frac{1}{(\nu+1)^{h+1}}$$

$$= O\left(\frac{k+1}{\nabla^{h-1} p_k}\right) \frac{1}{(k+1)^{h+1}} + O(1)\left(\frac{1}{\nabla^{h-1} p_k}\right) \frac{1}{(k+1)^h}$$

$$= O(1/P_k)$$

by Lemma 7(v), (vi) and Lemma 8 (iv).

If $h = 0$, then we split the sigma as before and obtain

$$\sum_{\nu=k}^{\infty} \frac{c_{\nu-k}^{(1)}}{(\nu+1)} \leqslant \frac{c_k^{(2)}}{k+1} + \sum_{\nu=2k+1}^{\infty} \frac{1}{(\nu+1)\,P_{\nu-k}}$$

$$= O(1/P_k)$$

by Lemma 7 (v), (vi) and hypothesis (3.6).

This proves (a). For the proof of (b), we proceed as before and obtain

$$\sum_{\nu=k}^{\infty} \frac{c_{\nu\ k}^{(h+1)}}{(\nu+1)^h \lambda_\nu} = O\left(\frac{1}{\lambda_k P_k}\right) + O\left(\frac{1}{P_k}\right) \sum_{\nu=2k+1}^{\infty} \frac{1}{(\nu+1)\lambda_\nu}$$

$$= O\left(\frac{1}{\lambda_k P_k}\right)$$

by (3.2).

Lemma 10—Let h be a non-negative integer such that $\nabla^h p_n \in \mathfrak{M}$. Let $\lambda_n > 0$ non-decreasing. Then

$$\text{(i)} \quad \sum_{n=k}^{\infty} \frac{\Delta_k^r\ p_{n-k}}{P_n} = O\left(\frac{1}{(k+1)^r}\right) \text{ for } r = 1, 2, \ldots h;$$

$$\text{(ii)} \quad \sum_{n=k}^{\infty} \frac{\Delta_k^r\ p_{n-k}}{(n+1)P_n} = O\left(\frac{1}{(k+1)^{r+1}}\right) \text{ for } r = 0, 1, 2, \ldots h;$$

$$\text{(iii)} \quad \sum_{n=k}^{\infty} \frac{\Delta_k^r\ p_{n-k}}{\lambda_n P_n} = O\left(\frac{1}{(k+1)^r \lambda_k}\right) \text{ for } r = 1, 2, \ldots h.$$

The result (iii) remains valid for $r = 0$ with the additional restriction contained in (3.2) [see Lemma 4].

Proof : By Lemma 8 (i), (iii)

$$\sum_{n=k}^{\infty} \frac{\Delta_k^r\ p_{n-k}}{P_n} = \left(\sum_{n=k}^{2k} + \sum_{n=2k+1}^{\infty}\right) \frac{\Delta_k^r\ p_{n-k}}{P_n}$$

$$\leqslant \frac{1}{P_k} \sum_{n=k}^{2k} \Delta_k^r\ p_{n-k} + K \sum_{n=2k+1}^{\infty} \frac{P_{n-k}}{(n-k)^{r+1}\ P_n}$$

$$\leqslant \frac{\nabla^{r-1}\ p_k}{P_k} + K \sum_{n=2k+1}^{\infty} \frac{1}{(n-k)^{r+1}}$$

$$< \frac{K}{(k+1)^r}.$$

This complets the proof of (i). The results (ii) and (iii) are immediate corollaries of (i) for $r \geqslant 1$. When $r = 0$, the result (ii) can be proved as in Lemma 4 by use of Lemma 8 (iii).

Lemma 11—Let h be a non-negative integer and let $\nabla^h p_n \in \mathfrak{M}$. Then

$$\sum_{n=k}^{\infty} | \Delta_k^r \Omega(n, k) | = O\left(\frac{1}{(k+1)^r}\right)$$

for $r = 0, 1, 2, \ldots h$.

PROOF: The case $r = 0$ is (4.10) which has been proved earlier in the proof of Lemma 8 (vi). Consider the case $r \geqslant 1$.

Since

$$\Delta_k^r \Omega(n, k) = \frac{\Delta_k^{r-1} p_{n-k}}{P_n} - \frac{\Delta_k^{r-1} p_{n-k-1}}{P_{n-1}}$$

$$= \frac{\Delta_k^r p_{n-k}}{P_n} - \frac{p_n}{P_n P_{n-1}} \Delta_k^{r-1} p_{n-k-1}$$

we obtain :

$$\sum_{n=k}^{\infty} | \Delta^r \Omega(n, k) | \leqslant \sum_{n=k}^{\infty} \frac{\Delta_k^r p_{n-k}}{P_n} + \sum_{n=k}^{\infty} \frac{p_n}{P_n P_{n-1}} \Delta_k^{r-1} p_{n-k-1}$$

$$= O\left(\frac{1}{(k+1)^r}\right) + O(1) \sum_{n=k}^{\infty} \frac{\Delta_k^{r-1} p_{n-k-1}}{(n+1) P_n}$$

$$= O\left(\frac{1}{(k+1)^r}\right)$$

by Lemma 8 (iii) and Lemma 10 (ii).

Lemma 12—Let h be a non-negative integer, $\nabla^h p_n \in \mathfrak{M}$ and let $\lambda_n > 0$ be such that λ_n increases to ∞ and

$$n(\lambda_{n+1} - \lambda_n) = O(\lambda_n).$$

Then

$$g_k = \sum_{n=k}^{\infty} \frac{(\lambda_{n+h+1} - \lambda_{k+h+1})}{\lambda_{n+h+1}} \frac{\Delta_k^{h+1} p_{n-k}}{P_n} = O\left(\frac{1}{(k+1)^{h+1}}\right).$$

$$h_k = \sum_{n=k}^{\infty} \frac{(\lambda_{n+h+1} - \lambda_{k+h+1})}{\lambda_{n+h+1}} \frac{\Delta_k^{h} p_{n-k-1}}{n P_n} = O\left(\frac{1}{(k+1)^{h+1}}\right).$$

Proof : Write $q_n = \nabla^h p_n$. Now by Abel's transformation

$$0 \leqslant \beta\,(n, k) = \sum_{\mu=k+1}^{n} (\lambda_{\mu+h+1} - \lambda_{k+h+1})\,(q_{\mu-k-1} - q_{\mu-k})$$

$$= - \sum_{\mu=k}^{n} \Delta_\mu\,(\lambda_{\mu+h+1})\,q_{\mu-k} - (\lambda_{n+h+2} - \lambda_{k+h+1})\,q_{n-k}$$

$$= \sum_{\mu=k}^{n} (\lambda_{\mu+h+2} - \lambda_{\mu+h+1})\,q_{\mu-k} - (\lambda_{n+h+2} - \lambda_{k+h+1})\,q_{n-k}$$

$$< \sum_{\mu=k}^{n} (\lambda_{\mu+h+2} - \lambda_{\mu+h+1})\,q_{\mu-k} \qquad \ldots(4.11)$$

as $q_n > 0$ and non-increasing.

As $(R, \lambda_n, 1)$ is regular, it follows that

$$R_n = \frac{1}{\lambda_{n+h+1}} \sum_{\mu=k}^{n} (\lambda_{\mu+h+2} - \lambda_{\mu+h+1})\,q_{\mu-k}$$

converges to a limit as $n \to \infty$. Now since $P_n \to \infty$ (by Lemma 8 (i)), we have from (4.11) that

$$\frac{\beta\,(n, k)}{\lambda_{n+h+1}\,P_n} \leqslant \frac{R_n}{P_n} = o\,(1) \qquad \ldots(4.12)$$

as $n \to \infty$, for any fixed k.

Now by Abel's transformation, (4.11) and (4.12), we have

$$g_k = \sum_{n=k+1}^{\infty} \Delta_n \left(\frac{1}{\lambda_{n+h+1}\,P_n} \right) \beta\,(n, k)$$

$$\leqslant \sum_{\mu=k}^{\infty} (\lambda_{\mu+h+2} - \lambda_{\mu+h+1})\,q_{\mu-k} \sum_{n=\mu}^{\infty} \Delta_n \left(\frac{1}{\lambda_{n+h+1}\,P_n} \right)$$

$$\leqslant \sum_{\mu=k}^{\infty} \frac{(\lambda_{\mu+h+2} - \lambda_{\mu+h+1})\,q_{\mu-k}}{\lambda_{\mu+h+1}\,P_\mu}$$

$$\leqslant K \sum_{\mu=k}^{\infty} \frac{q_{\mu-k}}{\mu\,P_\mu} \leqslant \frac{K}{(k+1)^{h+1}}$$

by Lemma 10 (ii).

Next,

$$h_k \leqslant \sum_{n=k+1}^{\infty} \frac{\Delta_k^h \; p_{n-k-1}}{n \; P_n} = O\left(\frac{1}{(k+1)^{h+1}} \right)$$

by Lemma (10) (ii).

This completes the proof of Lemma 12.

Lemma 13—Let h be any non-negative integer such that $\nabla^h p_n \in \mathcal{M}$ and let $\lambda_n > 0$ non-decreasing and

$$n \left(\lambda_{n+1} - \lambda_n \right) = O\left(\lambda_n \right).$$

Then

$$S_k = \sum_{n>k} \left(\frac{1}{\lambda_{k+h+1}} - \frac{1}{\lambda_{n+h+1}} \right) \left| \Delta_k^{h+1} \; \Omega\left(n, k\right) \right|$$

$$= O\left(\frac{1}{(k+1)^{h+1} \; \lambda_k} \right).$$

PROOF : Since

$$\Delta_k^{h+1} \; \Omega\left(n, k\right) = \frac{\Delta_k^{h+1} \; p_{n-k}}{P_n} - \frac{p_n}{P_n \; P_{n-1}} \; \Delta_k^h \; p_{n-k-1}$$

and since by Lemma 8 (iii)

$$\frac{p_n}{P_{n-1}} = O\left(\frac{1}{n} \right).$$

It follows that

$$S_k \leqslant \frac{K}{\lambda_k} \left(g_k + h_k \right)$$

$$\leqslant \frac{K}{(k+1)^{h+1} \; \lambda_k}$$

by Lemma 12.

This completes the proof.

Lemma 14—Let $\lambda_n > 0$ and non-decreasing. Let h be a non-negative integer. Then

$$n^{h+1} \; \lambda_n \; \Delta^{h+1} \left(\frac{1}{\lambda_n} \right) = O\left(1\right)$$

implies that

$$n^{r+1} \lambda_n \Delta^{r+1}\left(\frac{1}{\lambda_n} \right) = O(1) \text{ for } r = 0, 1, 2, \ldots h - 1.$$

PROOF : This can be proved as in Chow[2] (Lemma 13).

5. PROOF OF THEOREM 4

By Lemma 5 (a), it is enough to prove that J_k is bounded. By effecting $h + 1$ times successive Abel's transformation, we obtain :

$$\sum_{\nu=k}^{n} \frac{1}{\lambda_\nu} \Omega(n, \nu)\, c_{\nu\, k}$$

$$= \sum_{\nu=k}^{n} \Delta_\nu^{h+1} \left(\frac{\Omega(n, \nu)}{\lambda_\nu} \right) c_{\nu-k}^{(h+1)}$$

$$= \sum_{r=0}^{h+1} \binom{h+1}{r} \sum_{\nu=k}^{n} \left(\Delta_\nu^{h+1-r} \frac{1}{\lambda_{\nu+r}} \right) \Delta_\nu^r\, \Omega(n, \nu)\, c_{\nu-k}^{(h+1)} .$$

We now consider the terms $r = 0$ to $r = h$ together and the term $r = h + 1$ separately in the last sigma and obtain

$$J_k \leqslant J_k^{(1)} + J_k^{(2)}$$

where

$$J_k^{(1)} = \lambda_k P_k \sum_{n=k}^{\infty} \left| \sum_{\nu=k}^{n} \frac{1}{\lambda_{\nu\, h+1}} \left(\Delta_\nu^{h+1}\, \Omega(n, \nu)\, c_{\nu-k}^{(h+1)} \right| \right.$$

$$J_k^{(2)} = \sum_{r=0}^{h} \binom{h+1}{r} \lambda_k P_k \sum_{n=k}^{\infty} \left| \sum_{\nu=k}^{n} \Delta_\nu^{h+1-r} \left(\frac{1}{\lambda_{\nu+1r}} \right) \right.$$

$$\left. \times\, \Delta_\nu^r\, \Omega(n, \nu)\, c_{\nu-k}^{(h+1)} \right| .$$

Writting

$$\frac{1}{\lambda_{\nu+h+1}} = \frac{1}{\lambda_{\nu+h+1}} - \frac{1}{\lambda_{n+h+1}} + \frac{1}{\lambda_{n+h+1}}$$

we obtain

$$J_k^{(1)} \leqslant J_k^{(11)} + J_k^{(12)}$$

where

$$J_k^{(11)} = \lambda_k P_k \sum_{n=k}^{\infty} \Big| \sum_{\nu=k}^{n} \left(\frac{1}{\lambda_{\nu+h+1}} - \frac{1}{\lambda_{n+h+1}} \right) \left(\Delta_\nu^{h+1} \Omega(n, \nu) \right)$$

$$\times \; c_{\nu-k}^{(h+1)} \Big|$$

$$= O(1) \lambda_k P_k \sum_{n=\nu}^{\infty} c_{\nu-k}^{(h+1)} \sum_{n=\nu}^{\infty} \left(\frac{1}{\lambda_{\nu+h+1}} - \frac{1}{\lambda_{n+h+1}} \right)$$

$$\times \; | \Delta_\nu^{h+1} \Omega(n, \nu) |$$

$$= O(1) \lambda_k P_k \sum_{\nu=k}^{\infty} \frac{c_{\nu-k}^{(h+1)}}{(\nu+1)^{h+1} \lambda_\nu} = O(1).$$

by Lemmas 13 and 9. And

$$J_k^{(12)} = \lambda_k P_k \sum_{n=k}^{\infty} \frac{1}{\lambda_{n+h+1}} \; \Big| \sum_{\nu=k}^{n} \Delta_\nu^{h+1} \Omega(n, \nu)\, c_{\nu-k}^{(h+1)} \Big| .$$

It follows from (4.5) that

$$\sum_{\nu=k}^{n} c_{\nu-k}^{(h+1)} \; \Delta_\nu^{h+1} \; P_{n-\nu} = 1 \;\; (n \geqslant k)$$

and so

$$\sum_{\nu=k}^{n} c_{\nu-k}^{(h+1)} \; \Delta_\nu^{h+1} \; \Omega(n, \nu) = \frac{1}{P_n} - \frac{1}{P_{n-1}} .$$

Hence

$$J_k^{(12)} = O(1) \lambda_k P_k \sum_{n=k}^{\infty} \frac{1}{\lambda_{n+h+1}} \left(\frac{1}{P_{n-1}} - \frac{1}{P_n} \right)$$

(*equation continued on p.* 375)

$$= O(1)\, P_k \sum_{n=k}^{\infty} \left(\frac{1}{P_{n-1}} - \frac{1}{P_n} \right)$$

$$= O(1).$$

Therefore $J_k^{(1)}$ is bounded.

Lastly

$$J_k^{(2)} = \sum_{r=0}^{h} \binom{h+1}{r} \lambda_k\, P_k \sum_{\upsilon=k}^{\infty} c_{\nu-k}^{(h+1)} \left| \Delta_\nu^{h+1-r} \left(\frac{1}{\lambda_{\nu+r}} \right) \right|$$

$$\times \left| \sum_{n=\upsilon}^{\infty} \left| \Delta_\upsilon^{r}\, \Omega(n, \nu) \right| \right|$$

$$= O(1)\, \lambda_k\, P_k \sum_{\nu=k}^{\infty} c_{\nu-k}^{(h+1)} \frac{1}{(\nu+1)^r} \left| \Delta_\nu^{h+1-r} \left(\frac{1}{\lambda_{\nu+r}} \right) \right|$$

$$= O(1)\, \lambda_k\, P_k \sum_{\upsilon=k}^{\infty} \frac{c_{\upsilon-k}^{(h+1)}}{(\nu+1)^{h+1}\, \lambda_\nu} = O(1)$$

by hypotheses and Lemmas 11, 14 and 9.

This completes the proof of Theorem 4.

6. Further Lemmas

We need the following additional lemmas for the proofs of Theorems 5 and 6.

Lemma 15—Let h be a non-negative integer and let $\nabla^h p_n \in \mathfrak{M}$.

Then

$$M_k = \sum_{n=k}^{\infty} \frac{\left(\Delta_k^h\, p_{n-k} \right) c_{n-k}^{(h+2)}}{(n+h)\, P_n} = O\left(\frac{1}{P_k} \right).$$

Proof : Since $c_n^{(h+2)}$ is non-decreasing we obtain :

$$M_k = \left(\sum_{n=k}^{2k} + \sum_{n=2k+1}^{\infty} \right) \frac{\left(\Delta_k^h\, p_{n-k} \right) c_{n-k}^{(h+2)}}{(n+h)\, P_n}$$

(equation continued on p. 376)

$$\leqslant \frac{c_k^{(h+2)}}{(k+h)\,P_k} \sum_{n=k}^{2k} \Delta_k^h\, p_{n-k} + K \sum_{n=2k+1}^{\infty}$$

$$\times \frac{(n-k+1)\,\Delta_k^h\, p_{n-k}}{(n+h)\,P_n\,\Delta_k^{h-1}\, p_{n-k}}$$

$$\leqslant \frac{c_k^{(h+2)}\,\nabla^{h-1} p_k}{(k+h)\,P_k} + K \sum_{n=2k+1}^{\infty} \frac{(n-k+1)\,p_{n-k}}{(n+h)\,P_n\,P_{n-k}}$$

$$\leqslant \frac{K}{P_k} + \sum_{n=k+1}^{\infty} \left(\frac{1}{P_{n-k-1}} - \frac{1}{P_{n-k}} \right) \leqslant \frac{K}{P_k}$$

by hypotheses, Lemma 7 (vi) and Lemma 8 (v).

This completes the proof.

Lemma 16—Let h be a non-negative integer and let (λ_n) be a positive non-decreasing sequence such that

$$n^{h+1}\,\Delta^{h+1}\,\lambda_n = O\,(\lambda_n).$$

Then

$$n^{r+1}\,\Delta^{r+1}\,\lambda_n = O\,(\lambda_n)$$

for

$$r = 0, 1, 2, \ldots h-1.$$

Proof : It is enough to prove the result for $r = h-1$, since the general result can then be obtained by repeated applications of this case. The proof is based on one verson of the 'discrete' analogue of Taylor's theorem with remainder and is given by the following formula, valid for $0 \leqslant k \leqslant n$:

$$\lambda_{n-k} = \sum_{r=0}^{h} \binom{k+r-1}{r} \Delta^r \lambda_n + \sum_{\nu=n-k}^{n-1} \binom{\nu+h-n+k}{h} \Delta^{h+1} \lambda_\nu. \qquad \ldots(6.1)$$

This is easily proved by induction on $h = 0$; and if in the second term in (6.1), we substitute

$$\Delta^{h+1}\,\lambda_\nu = \Delta^{h+1}\,\lambda_n + \sum_{\mu=\nu}^{n-1} \Delta^{h+2}\,\lambda_\mu$$

we obtain, after simplification, the same formula as (6.1) but with h replaced by $h + 1$. This establishes (6.1).

Note that if c is a constant such that $0 < c < 1$, then for $0 \leqslant k \leqslant cn$ and $n-k \leqslant \nu \leqslant n - 1$, we have $\nu > \alpha n$ where $\alpha > 1 - c > 0$. Hence

$$\Delta^{h+1} \lambda_\nu = O\left(\frac{\lambda_\nu}{\nu^{h+1}}\right) = O\left(\frac{\lambda_\nu}{n^{h+1}}\right) = O\left(\frac{\lambda_n}{n^{h+1}}\right)$$

as (λ_ν) is non-decreasing. Since the sum of the coefficients in the second sum in (6.1) is

$$\binom{k+h}{h+1} = O\left(k^{h+1}\right) = O\left(n^{h+1}\right)$$

it follows that, uniformly in $0 \leqslant k \leqslant cn$, we have

$$I\ (n, k) = \sum_{r=0}^{h} \binom{k+r-1}{r} \Delta^r \lambda_n = O\left(\lambda_n\right). \qquad \ldots(6.2)$$

Now let d be a constant with $0 < d < \frac{1}{h}$, and let $k = [nd]$. It follows from (6.2) that

$$\sum_{\nu=0}^{h} (-1)^\nu \binom{h}{\nu} I\left(n, \nu k\right) = O\left(\lambda_n\right) \qquad \ldots(6.3)$$

since the coefficients are constants. Now the sum in the left of (6.3) is the hth difference of $I\ (n, \nu k)$ (regarded as a function of ν) taken for $\nu = 0$. The terms with $r < h$ in the sum defining $I\ (n, \nu k)$ are polynomials in ν of degree less than h; their hth differences are therefore 0. The terms with $r = h$ is a polynomial of degree h, the coefficient of ν^h being

$$(-1)^h \frac{k^h}{h!} \Delta^h \lambda_n$$

its hth diflerence is therefore

$$k^h \Delta^h \lambda_n.$$

Hence (6.3) reduces to

$$\Delta^h \lambda_n = O\left(\frac{\lambda_n}{k^h}\right) = O\left(\frac{\lambda_n}{n^h}\right)$$

by definition of k. Hence the result.

Lemma 17—Let $\lambda_n > 0$ and non-decreasing. Then the hypotheses (3.4) and (3.5) are equivalent; i.e.

$$n^{h+1} \lambda_n \Delta^{h+1} \left(\frac{1}{\lambda_n}\right) = O\left(1\right)$$

if and only if

$$n^{h+1}\,\Delta^{h+1}\,\lambda_n = O\,(\lambda_n).$$

PROOF : Suppose that (3.5) holds. Then by Lemma 16

$$n^{r+1}\,\Delta^{r+1}\,\lambda_n = O\,(\lambda_n)$$

for

$$r = 0,\,1,\,2,\,\ldots\,h-1.$$

Write $\lambda_n^{(a)}$ for any product of terms each of the form λ_{n+b} where b is a constant integer. The values of b for different factors of the product may not be the same. We use a similar notation for $(\Delta^r\,\lambda_n)^{(a)}$. When $a = 0$, this is taken as meaning 1.

Since

$$\Delta\left(\frac{1}{\lambda_n}\right) = -\,\frac{\Delta\,\lambda_n}{\lambda_n\,\lambda_{n+1}}$$

we can verify by induction on h that $\Delta^{h+1}\left(\dfrac{1}{\lambda_n}\right)$ is the sum of a finite number of terms of the form :

$$\frac{(\Delta\lambda_n)^{(a_1)}\;(\Delta^2\,\lambda_n)^{(a_2)}\,..\;(\Delta^{h+1}\,\lambda_n)^{(a_{n+1})}}{\lambda_n^{(b)}}$$

where $a_1, a_2, \ldots,$ are non-negative integers such that

$$a_1 + 2a_2 + 3a_3 + \ldots + (h+1)\,a_{h+1} = h+1$$

$$a_1 + a_2 + \ldots + a_{h+1} + 1 = b.$$

It follows easily that (3.4) holds.

The converse implication may be proved in a similar way using the formula just stated with λ_n replaced by $1/\lambda_n$.

7. PROOF OF THEOREM 5

By Lemma 5 (b), it is enough to prove that I_k is bounded. Now

$$I_k < I_k^{(1)} + I_k^{(2)}$$

where

$$I_k^{(1)} = \sum_{n=k}^{\infty} \frac{1}{\lambda_n\,P_n}\;\left|\sum_{\mu=0}^{k-1} p_\mu \sum_{v=k}^{n} c_{v-\mu}\,p_{n-v}\,\lambda_v\right|$$

$$I_k^{(2)} = P_{k-1} \sum_{n=k}^{\infty} \frac{1}{\lambda_n P_n} \mid \sum_{\nu=k}^{n} \lambda_\nu p_{n-\nu} c_{\nu-k} \mid .$$

By Abel's transformation h times.

$$\sum_{\nu=k}^{n} \lambda_\nu p_{n-\nu} c_{\nu-\mu}$$

$$= \sum_{\nu=k}^{n} \Delta_\nu(\lambda_\nu p_{n-\nu}) c^{(1)}_{\nu-\mu} - c^{(1)}_{k-\mu-1} \lambda_k p_{n-k}$$

$$= \sum_{\nu=k}^{n} \Delta^h_\nu (\lambda_\nu p_{n-\nu}) \Big(c^{(h)}_{\nu-\mu} - \sum_{r=1}^{h} c^{(h)}_{k-\mu-1} \Delta^{r-1}_k (\lambda_k p_{n-k}).$$

Hence

$$I_k^{(1)} \leqslant I_k^{(11)} + I_k^{(12)}$$

where

$$I_k^{(11)} = \sum_{r=0}^{h} \binom{h}{r} \sum_{\mu=0}^{k-1} p_\mu \sum_{n=k}^{\infty} \frac{1}{\lambda_n P_n} \mid \sum_{\nu=k}^{n} (\Delta^{h-r} \lambda_{\nu+r})$$

$$(\Delta^r_\nu p_{n-\nu}) c^{(h)}_{\nu-\mu} \mid$$

$$< \sum_{r=0}^{h} \binom{h}{r} \sum_{\mu=0}^{k-1} p_\mu \sum_{\nu=k}^{\infty} (\Delta^{h-r} \lambda_{\nu+r}) \mid c^{(h)}_{\nu-\mu} \mid \sum_{n=\nu}^{\infty} \mid \frac{\Delta^r_\nu p_{n-\nu}}{\lambda_n P_n} \mid$$

$$= O(1) \sum_{r=0}^{h} \binom{h}{r} \sum_{\mu=0}^{h-1} p_\mu \sum_{\nu=k}^{\infty} \frac{\mid c^{(h)}_{\nu-\mu} \mid}{\nu^r} \frac{\mid \Delta^{h-r}_\nu \lambda_{\nu+r} \mid}{\lambda_\nu}$$

$$= O(1) \sum_{\mu=0}^{k-1} p_\mu \sum_{\nu=k}^{\infty} \frac{\mid c^{(h)}_{\nu-\mu} \mid}{\nu^r} \cdot \frac{1}{\nu^{h-r}}$$

$$= O(1) \frac{1}{k^h} \sum_{\mu=0}^{k-1} p_\mu \sum_{\nu=k}^{\infty} \mid c^{(h)}_{r-\mu} \mid$$

(equation continued on p. 380)

$$= O(1) \frac{1}{k^h} \sum_{\mu=0}^{k-1} p_\mu \, c_{k-\mu-1}^{(h+1)}$$

$$= O(1). \frac{1}{k^h} k^h = O(1)$$

by Lemma 8 (iii), hypothesis (3.5), Lemma 7 (iv) and by the fact that

$$\sum_{k=0}^{n} p_{n-k} \, c_k^{(h+1)} = A_n^h$$

which, by virtue of (1.21), follows from the identity :

$$(\Sigma \, p_n \, x^n) \, (\Sigma \, c_n^{(h+1)} \, x^n) = \Sigma \, A_n^h \, x^n. \qquad \text{...(7.1)}$$

Before we consider $I_k^{(12)}$, note that by hypothesis (3.5) and Lemma 16, we have

$$n\,(\lambda_n - \lambda_{n+1}) = O(\lambda_n)$$

and this implies that

$$\lambda_n \sim \lambda_{n+1}.$$

Hence

$$\frac{\lambda_{n+\theta}}{\lambda_n} \to 1 \qquad \text{...(7.2)}$$

as $n \to \infty$, for fixed θ.

Now

$$I_k^{(12)} = \sum_{n=k}^{\infty} \frac{1}{\lambda_n P_n} \left| \sum_{\mu=0}^{k-1} p_\mu \sum_{r=1}^{h} c_{k-\mu-1}^{(r)} \, \Delta_k^{r-1} (\lambda_k \, p_{n-k}) \right|$$

$$= \sum_{n=k}^{\infty} \frac{1}{\lambda_n P_n} \left| \sum_{\mu=0}^{k-1} p_\mu \sum_{r=1}^{h} \sum_{\theta=0}^{r-1} \binom{r-1}{\theta} c_{k-\mu-1}^{(r)} \right.$$

$$\left. \times \left(\Delta_k^{r-1-\theta} (\lambda_{k+\theta}) \right) \Delta_\mu^\theta \, p_{n-\mu} \right|$$

$$\leqslant \sum_{r=1}^{h} \sum_{\theta=0}^{r-1} \binom{r-1}{\theta} \left| \sum_{n=k}^{\infty} \frac{1}{\lambda_n P_n} \left(\Delta_k^{r-1-\theta} \lambda_{k+\theta} \right) \right.$$

(equation continued on p. 381)

$$\times \ (\Delta_k^{\theta} \ p_{n-k}) \sum_{\mu=0}^{k} p_\mu \ c_{k-\mu-1}^{(r)} \ |$$

$$= O(1) \sum_{r=1}^{h} \sum_{\theta=0}^{r-1} \binom{r-1}{\theta} \sum_{n=k}^{\infty} \frac{1}{\lambda_n P_n} \ | \ \Delta_k^{r-1-\theta} \ \lambda_{k+\theta} \ |$$

$$\times \ (\Delta_k^{\theta} \ p_{n-k}) \, k^{r-1}$$

$$= O \ (1) \sum_{r=1}^{h} \sum_{\theta=0}^{r-1} \binom{r-1}{\theta} \ | \ \Delta_k^{r-1-\theta} \ \lambda_{k+\theta} \ | \ k^{r-1} \sum_{n=k}^{\infty}$$

$$\times \ \frac{\Delta_k^{\theta} \ p_{n-k}}{\lambda_n P_n}$$

$$= O(1) \sum_{r=1}^{h} \sum_{\theta=0}^{r-1} \binom{r-1}{\theta} \frac{k^{r-1}}{k^{\theta}} \ | \ \Delta_k^{r-1-\theta} \ \lambda_{k+\theta} \ | \ \frac{1}{\lambda_k}$$

$$= O(1) \sum_{r=1}^{h} \sum_{\theta=0}^{r-1} \binom{r-1}{\theta} \frac{\lambda_{k+\theta}}{\lambda_k} = O(1).$$

by Lemma 10 (ii), (7.2), the hypotheses, and the fact that for $r \geqslant 0$,

$$\sum_{k=0}^{n} p_k \ c_{n-k}^{(r)} = A_n^{r-1} .$$

(This identity follows from an identity similar to (7.1)).

Thus

$$I_k^{(1)} \leqslant I_k^{(11)} + I_k^{(12)} = O(1).$$

Now we consider $I_k^{(2)}$. By making $h + 1$ times Abel's transformation we have

$$\sum_{\nu=k}^{n} \lambda_\nu \, p_{n-\nu} \ c_{\nu-k}$$

$$\sum_{r=0}^{h} \binom{h}{r} = \sum_{\nu=k}^{n} \ (\Delta_\nu^{h-r+1} \ \lambda_{\nu+r}) \ (\Delta_\nu^{r} \ p_{n-\nu}) \ c_{\nu-k}^{(h+1)} \ \Big).$$

Hence

$$I_k^{(2)} = \sum_{r=0}^{h} \binom{h}{r} p_k \sum_{n=k}^{\infty} \frac{1}{\lambda_n P_n} \mid \sum_{v=k}^{n} (\Delta_v^{h-r+1} \lambda_{v+r})$$

$$\times (\Delta_v^r \; p_{n-v}) \; c_{v-k}^{(h+1)} \mid$$

$$\leqslant \sum_{r=0}^{h} \binom{h}{r} p_k \sum_{v=k}^{\infty} \mid (\Delta_v^{h-r+1} \lambda_{v+r} \mid c_{v-k}^{(h+1)} \sum_{n=v}^{\infty} \frac{\Delta_v^r p_{n-v}}{\lambda_n P_n}$$

$$= O(1) \sum_{r=0}^{h} \binom{h}{r} p_k \sum_{v=k}^{\infty} \frac{\mid \Delta_v^{h-r+1} \lambda_{v+r} \mid c_{v-k}^{(h+1)}}{v^r \lambda_v}$$

$$= O(1) \sum_{r=0}^{h} \binom{h}{r} p_k \sum_{v=k}^{\infty} \mid \frac{c_{v-k}^{(h+1)}}{v^{h+1}} = O(1)$$

by Lemma 10 (iii), Lemma 9 (a), Lemma 16 and hypotheses.

We note that the result of Lemma 9 (a) is valid with the assumption (3.6) in the case $h = 0$, and this is not needed in the case $h > 0$.

Now we consider $I_k^{(2)}$ in the case $h = 0$, under hypothesis (3.7) in place of (3.6). Now

$$\theta(n, k) = \sum_{v=k}^{n} \lambda_v \; p_{n-v} \, c_{v-k}$$

$$= \sum_{v=k}^{n} \Delta \lambda_v \sum_{\mu=k}^{v} p_{n-\mu} \; c_{\mu-k} + \lambda_{n+1} \; \delta_{n,k}$$

where

$$\delta_{nk} = \sum_{\mu=k}^{n} p_{n-\mu} \; c_{\mu-k} = \begin{cases} 1 \; (n = k) \\ O \; (n \neq k). \end{cases}$$

Now by hypothesis (3.5) and (3.7) (in the case $h = 0$) we obtain,

$$\Delta \lambda_v = O\left(\frac{\lambda_v}{v}\right) = O\left(\frac{\lambda_n}{n}\right).$$

Hence by Lemma 7 (vii)

$$\theta(n, k) = O(1) \; p_{n-k} \sum_{\nu=k}^{n} | \Delta \lambda_\nu | \, c_{\nu-k}^{(1)} + \lambda_{n+1} \, \delta_{nk}$$

$$= O(1) \, \frac{\lambda_n}{n} \, p_{n-k} \, c_{n-k}^{(2)} + \lambda_{n+1} \, \delta_{nk}.$$

Hence

$$I_k^{(2)} = O(1) \, P_k \sum_{n=k}^{\infty} \frac{p_{n-k} \, c_{n-k}^{(2)}}{n \, P_n} + \frac{P_k \, . \, \lambda_{k+1}}{\lambda_k \, . \, P_k}$$

$$= O(1)$$

by Lemma 15.

8. Proof of Theorem 6

This follows by combining Theorem 4 and Theorem 5, taking note of Lemma 17.

9. Corollaries

We know that Cesàro mean and the modified Cesàro mean with weight n are equal. In the following corollaries we examine special sequences λ for which Nörlund and modified Nörlund means are equivalent.

From Theorem 4, we obtain the following.

Corollary 1—Let $\nabla^h p_n \in \mathfrak{M}$ and let

$$\lambda_n = n^{\beta} (\log n)^{\delta}.$$

Then

$$| N', p, \lambda | \Rightarrow | N, p |$$

in the following cases :

(i) $h > 0$ and either $\beta > 0$, δ real or $\beta = 0$, $\delta > 0$.

(ii) $h = 0$, $\beta > 0$, δ real

In particular

$$| C', \alpha, \lambda | \Rightarrow | C, \alpha |$$

in the following cases :

(i) $\alpha > 1$ and either $\beta > 0$, δ real or $\beta = 0$, $\delta > 0$

(ii) $0 < \alpha \leqslant 1$, $\beta > 0$, δ real.

Similarly from Theorem 5, we obtain the following.

Corollary 2—Let $\nabla^h p_n \in \mathscr{M}$ and let

$$\lambda_n = n^\beta (\text{long } n)^\delta.$$

Then

$$| N, p | \Rightarrow | N', p, \lambda |$$

in the following cases :

(i) $h > 0$, $\beta > 0$, δ real

(ii) $h = 0$ and either $\beta \geqslant 1$, δ real or (3.6) holds. In particular

$$| C, \alpha | \Rightarrow | C', \alpha, \lambda |$$

if $\beta > 0$, δ real. And

$$| N, \frac{1}{n+1} | \Rightarrow | N', \frac{1}{n+1}, \lambda |$$

if $\beta \geqslant 1$, δ real.

Combining Corollaries 1 and 2, we obtain the following.

Corollary 3—Let $\nabla^h p_n \in \mathscr{M}$ and let

$$\lambda_n = n^\beta (\log n)^\delta.$$

Then

$$| N', p, \lambda | \sim | N, p |$$

in the following cases :

(i) $h > 0$, $\beta > 0$, δ real

(ii) $h = 0$ and either $\beta > 1$, δ real or (3.6) holds. In particular

$$| C', \alpha, \lambda | \sim | C, \alpha |$$

if $\beta > 0$, δ real and

$$| N, \frac{1}{n+1} | \sim | N', \frac{1}{n+1}, \lambda |$$

If $\beta \geqslant 1$, δ real.

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SCATTERING OF A COMPRESSIONAL WAVE AT THE CORNER OF A QUARTER SPACE

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The problem of scattering of a compressional wave at the corner of an elastic quarter space ($x \geqslant 0$, $z \geqslant 0$) has been discussed in the present paper. One face of the quarter space is free and other is assumed to be rigid not permitting the displacements across it. The technique is due to Wiener and Hopf. The scattered field possesses the character of transverse cylindrical waves. The numerical computation for the amplitude of the scattered field exhibits a sharp fall versus the small values of the wave number.

INTRODUCTION

A problem of special interest in seismology is the problem of scattering of an elastic wave at a corner of an elastic medium as it leads to a phenomena of a surface wave entering into another medium after travelling through a medium. Several authors[4–6,11] have studied the problem of Rayleigh wave propagation in elastic wedges both theoretically and experimentally. Recently, Momoi[7] has discussed the problem of scattering of a Rayleigh wave in an elastic quarter space. He has extended his study to Rayleigh wave scattering due to a rectangular mountain[8]. The three-dimensional problem of scattering of waves in an elastic quarter space has been considered recently by Gautesen[3]. Both Momoi[7,8] and Gautesen[3] have studied the problem of scattering of surface waves in an elastic quarter space using rigorous numerical computations to get approximate results. Deshwal and Mann[1] have attempted the problem of scattering of Rayleigh waves at the corner of an elastic quarter space to obtain exact results using the technique of Wiener and Hopf.

In the present paper, it is being contemplated to study the problem of scattering of a compressional wave at the corner of a quarter space. Most of the mountains are deep-rooted with their bases in the solid mantle of the earth. They are assumed to be rigid forming the rigid boundary of the present problem. They peep out of the earth and scatter the seismic waves. Physically, mountains within the earth form the rigid boundary and the surface of the earth is the free surface of the quarter space. The Fourier transformation and the function theoretic technique due to Wiener and Hopf is the method of solution.

BASIC EQUATIONS

The problem is two-dimensional. The waves propagate in the zx-plane. The x-axis is in the free surface and the z-axis being along the rigid boundary with origin at the corner of the quarter space $x \geqslant 0, z > 0$. The medium is homogeneous, isotropic and slightly dissipative. A time-harmonic two-dimensional compressional wave is incident at the corner (Fig. 1) and gives rise to the reflected and scattered waves.

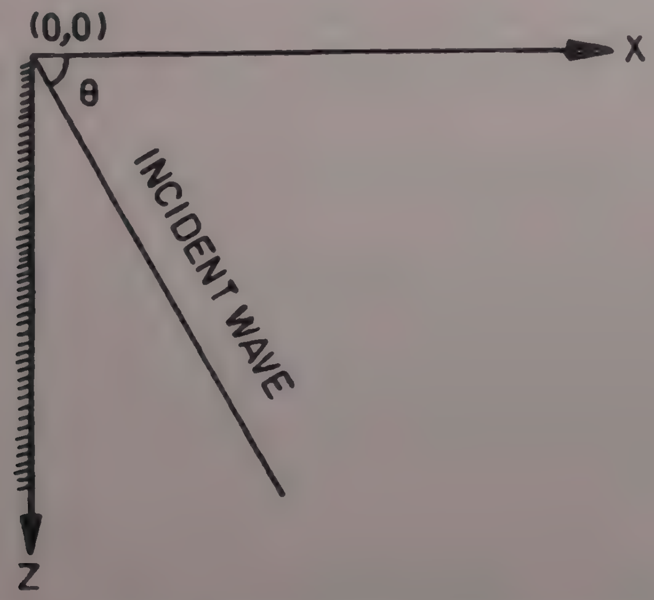

FIG. 1. A solid quarter space with a rigid boundary.

The incident wave is

$$\phi_i (x, z) = \exp [-ik (x \cos \theta + z \sin \theta)]. \quad \ldots(1)$$

The total potentials in the medium are

$$\phi_t (x, z) = \phi_i (x, z) + \phi (x, z) \quad \ldots(2)$$

$$\psi_t (x, z) = \psi (x, z). \quad \ldots(3)$$

The wave equations are

$$(\nabla^2 + k^2) \phi = 0 = (\nabla^2 + k'^2) \psi. \quad \ldots(4)$$

$k = k_1 + ik_2$, $k' = k'_1 + ik'_2$ are complex wave numbers with positive imaginary parts. The displacements (u, w) are given by

$$u = \frac{\partial \phi_t}{\partial x} - \frac{\partial \psi_t}{\partial z}, \; w = \frac{\partial \phi_t}{\partial z} + \frac{\partial \psi_t}{\partial x}. \quad \ldots(5)$$

BOUNDARY CONDITIONS

The conditions on the boundaries and at distant points of the quarter space

are

$$\text{(i) } u = 0 = w, \ x = 0, \ z \geqslant 0 \qquad \text{...(6)}$$

$$\text{(ii) } p_{zz} = 0 = p_{zx}, \ z = 0, \ x \geqslant 0 \qquad \text{...(7)}$$

where p_{zz}, p_{zx} are the normal and the shear stresses. The conditions (6) imply that ϕ_t and ψ_t satisfy the Cauchy-Riemann equations and are harmonic functions. Equations (4) will result in

$$\phi_t (x, z) = 0 = \psi_t (x, z), \ x = 0, \ z \geqslant 0. \qquad \text{...(8)}$$

The half-range Fourier transform

$$\bar{\phi}_+ (p, z) = \int_0^\infty \phi (x, z) \, e^{ipx} \, dx, \ p = \alpha + i\beta \qquad \text{...(9)}$$

is analytic along with its derivatives in the region $\beta > -d$ of the complex p-plane if for given z

$$| \phi (x, z) | \sim M \exp (-d \, | \, x \, |), \ M, d > 0. \qquad \text{...(10)}$$

The Fourier transform of $\psi (x, z)$ and its derivatives with respect to z are analytic in the same region.

Discussion of the Problem

Let us take the Fourier transform of the first of eqns. (4) to obtain

$$\left(\frac{d^2}{dz^2} - Y^2 \right) \bar{\phi}_+ (p, z) = \left(\frac{\partial \phi}{\partial x} \right)_0 - ip \, (\phi)_0 \qquad \text{...(11)}$$

where $Y = \pm (p^2 - k^2)^{1/2}$. The sign before the radical is such that the real part of Y is always positive for all p. The subscript 0 denotes the value at $x = 0$. The first of the conditions (8) simplifies to

$$(\phi)_0 = - (\phi_i)_0 = - \exp (-ik \, z \sin \theta). \qquad \text{...(12)}$$

A complete solution of (11), which holds when $z \to \infty$, is

$$\bar{\phi}_+ (p,z) - \bar{\phi}_+ (-p, z) = A (p) \, e^{-Yz} - \frac{2ip \exp (-ik \, z \sin \theta)}{p^2 - k^2 \cos^2 \theta} \qquad \text{...(13)}$$

Putting $z = 0$ in (13) and in its derivative with respect to z and eliminating $A (p)$ between the resulting equations to obtain

$$\bar{\phi}_+ (p) - \bar{\phi}_+ (-p) = - \left[\bar{\phi}'_+ (p) - \bar{\phi}'_+ (-p) + \frac{2pk \sin \theta}{p^2 - k^2 \cos^2 \theta} \right] - \frac{2ip}{p^2 - k^2 \cos^2 \theta}. \qquad \text{...(14)}$$

The notations $\bar{\phi}_+(p)$, $\bar{\phi}'_+(p)$ are used for $\bar{\phi}_+(p, 0)$, $\bar{\phi}'_+(p, 0)$ etc. Equation (14) is a Wiener – Hopf type functional equation to be solved for two unknown functions $\bar{\phi}_+(p)$ and $\bar{\phi}_+(-p)$.

Solution of the Wiener-Hopf Equation

Equation (14) can be written as

$$\bar{\phi}_+(p) + \frac{\bar{\phi}'_+(p)}{Y} + \frac{iY + k\sin\theta}{Y(p+k\cos\theta)} = \bar{\phi}_+(-p) + \frac{\bar{\phi}'_+(-p)}{Y} - \frac{iY + k\sin\theta}{Y(p-k\cos\theta)} \qquad \text{...(15)}$$

The left-hand member is analytic in the region $\beta > -k_2\cos\theta$ and the right hand member in $\beta < k_2\cos\theta$. By analytic continuation, they represent an entire function. The points $p = \pm k$ are excluded by the branch cuts. Each member tends to zero as $|p| \to \infty$. By an extension of Liouville theorem each member is identically zero. Hence

$$\bar{\phi}_+(p) = -\frac{\bar{\phi}'_+(p)}{Y} - \frac{iY + k\sin\theta}{Y(p+k\cos\theta)},\ p \neq \pm k\cos\theta. \qquad \text{..(16)}$$

Similarly

$$\psi_+(p) = -\psi'_+(p)/\delta,\ \delta = \pm(p^2 - k'^2)^{1/2} \qquad \text{...(17)}$$

the choice of signs for δ is same as for Y.

Taking the Fourier transforms of the conditions (7) and using (16) and (17), it is obtained that

$$-2ip\,\bar{\phi}'_+(p) + \frac{(2p^2 - k'^2)\,\psi'_+(p)}{\delta} = \frac{2ipk\sin\theta}{p-k\cos\theta} \qquad \text{...(18)}$$

$$\frac{(2p^2 - k'^2)\,\bar{\phi}'_+(p)}{Y} + 2ip\,\psi'_+(p) = \left[\frac{k\sin\theta + iY}{Y(p+k\cos\theta)} + \frac{i}{p-k\cos\theta}\right](2p^2 - k'^2). \qquad \text{...(19)}$$

These equations are solved for $\bar{\psi}'_+(p)$. p is changed to $-p$ to find the value of $\bar{\psi}'_+(p) - \bar{\psi}'_+(-p)$. Then

$$\psi_+(p, z) - \psi_+(-p, z) = -\frac{1}{\delta}\Big[\psi'_+(p) - \psi'_+(-p)\Big]\exp(-\delta z)$$

$$= \frac{8p\mu k \cos\theta\,(2p^2 - k'^2)(Y - ik\sin\theta)}{(p^2 - k^2\cos^2\theta)\,F(p)} \cdot e^{-\delta z} \qquad \ldots(20)$$

where

$$F(p) = \mu\,[(2p^2 - k'^2)^2 - 4p^2 Y\delta]. \qquad \ldots(21)$$

VARIOUS WAVES

The potential function $\psi(x, z)$ is given by the inverse Fourier transform

$$\psi(x, z) = \frac{1}{2\pi}\int_{-\infty+i\beta}^{\infty+i\beta} \bar{\psi}_+(p, z)\, e^{-ipx}\, dp,\; x > 0 \qquad \ldots(22)$$

where $-d < \beta < d$. The factor $\exp(-ipx) = \exp(i\alpha x)\exp(\beta x)$ in (22) vanishes as $p \to -\infty$ in the lower part of the complex plane if $x > 0$. If the contour of integration is chosen in the lower half of the complex plane, where $\psi_+(-p, z)$ is analytic, then

$$\psi(x, z) = \frac{1}{2\pi}\int_{-\infty+i\beta}^{\infty+i\beta} [\bar{\psi}_+(p, z) - \bar{\psi}_+(-p, z)]\, e^{-ipx}\, dp. \qquad \ldots(23)$$

The value of expression within the brackets is obtained in (20). (23) is evaluated along a closed contour in the lower half of the complex plane with $p = -k, -k'$ as the branch points, indentations around $p = \pm k\cos\theta$ and satisfying the conditions $\mathrm{Re}(Y) = 0 = \mathrm{Re}(\delta)$ (Fig. 2). The conditions according to Ewing *et al.*[2] imply hyperbolic paths for the branch cuts at $p = -k, -k'$. Indentations around $p = \pm k\cos\theta$ contribute the reflected transverse waves

$$D(k)\exp(-ikx\cos\theta - \delta' z) \text{ and } -D(k)\exp(ikx\cos\theta - \delta' z) \qquad \ldots(24)$$

where

$$D(k) = 4\mu k^2 \sin\theta\cos\theta\,(2k^2\cos^2\theta - k'^2)/F(k\cos\theta) \qquad \ldots(25)$$

and $\delta' = (k^2\cos^2\theta - k'^2)^{1/2}$. These waves cancel each other on the rigid boundary. The compressional waves reflected from the boundaries are

$$-\tfrac{1}{2}\exp[-ik(x\cos\theta\; z\sin\theta)] \text{ and } -\tfrac{1}{2}\exp[ik(x\cos\theta - z\sin\theta)]. \qquad \ldots(26)$$

These waves together with the incident wave vanish on the rigid boundary satisfying the boundary conditions (8).

The scattered waves are the contributions of the integrals along the branch cuts. Along the branch cut at $p = -k$, real part of Y is zero and imaginary part of Y has

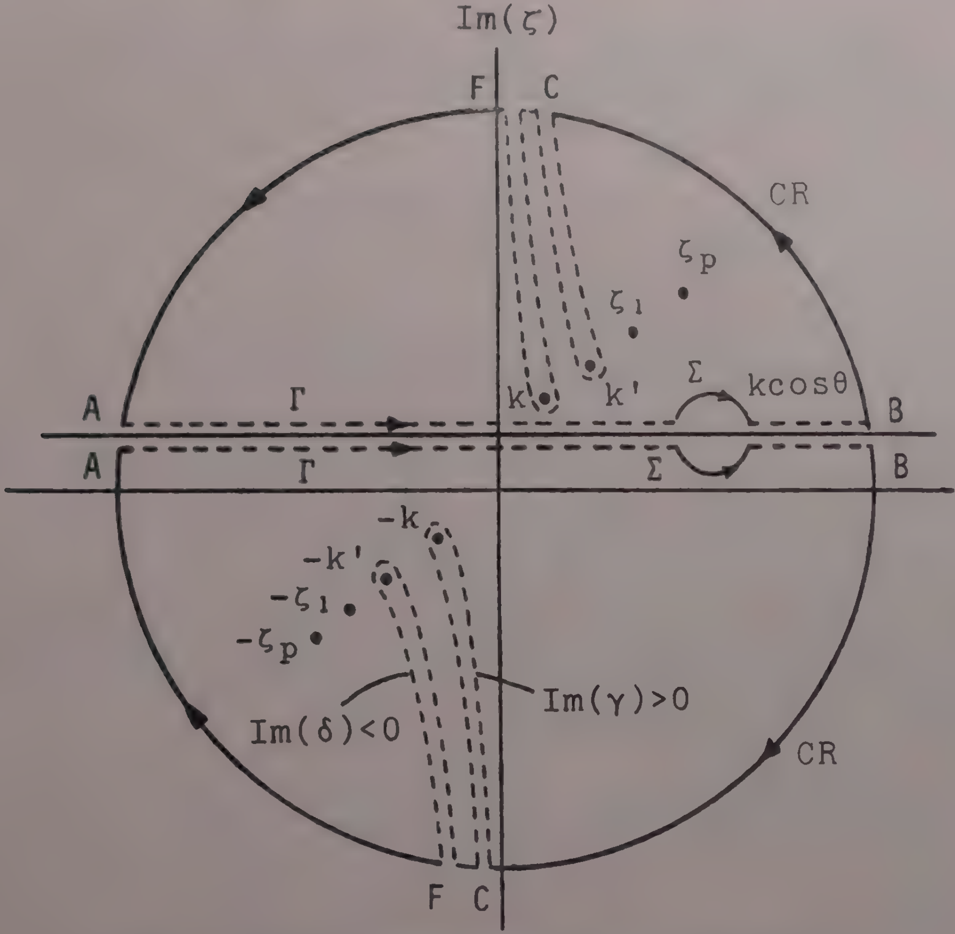

FIG. 2. Contour of integration in the complex P-plane.

opposite signs along the two sides of the branch cut. δ is unchanged along the cut. The integrals cancel each other along the branch cut at $p = -k$. Thus the scattered waves are not of the compressional type. For the contribution along the branch cut at $p = -k'$, let us take $p = -k' - iu$, u being small, then

$$\delta^2 = (-k' - iu)^2 - k'^2 = 2iu\,(k_1' + ik_2') - u^2 \qquad \ldots (27)$$

Along the cut, real $(\delta) = 0$ and δ^2 is real and negative. Thus $\delta = \pm i\delta'$, $\delta' = (2k_2' u)^{1/2}$, $k_1' = 0$ and u varies from 0 to ∞ along the cut. Evaluating (23) along the sides of the branch cut to find

$$\psi_1(x, z) = \frac{ie^{-k_2' x}}{2\pi} \int_0^\infty \Big[[\bar{\psi}_+(p, z) - \bar{\psi}_+(-p, z)]_{\delta = i\delta'}$$

$$- [\bar{\psi}_+(p, z) - \bar{\psi}_+(-p, z)]_{\delta = -i\delta'} \Big] e^{-ux}\, du \qquad \ldots (28)$$

$$\int_0^\infty \left[M(u) \sin\left(\sqrt{2k_2' u z}\right) + N(u)\sqrt{u} \cos\left(\sqrt{2k_2' u z}\right)\right]$$

$$e^{-(k_2' + u)x}\, du \qquad \ldots(29)$$

where

$$M(u) = \frac{8pk \cos\theta\,(Y - ik\sin\theta)\,\mu^2\,(2p^2 - k'^2)^3}{\pi\, H(p)\,(p^2 - k^2\cos^2\theta)} \qquad \ldots(30)$$

$$N(u) = -\frac{32\beta^3\, Yk\cos\theta\,(Y - ik\sin\theta)\,(2k_2)^{1/2}\,\mu^2\,(2p^2 - k'^2)}{\pi\, H(p)\,(p^2 - k^2\cos^2\theta} \qquad \ldots(31)$$

$$H(p) = \mu^2\,[(2p^2 - k'^2)^4 + 16p^4\, Y^2\, \delta^2]. \qquad \ldots(32)$$

Since u is small, $M(0)$ and $N(0)$ are retained. Following Laplace integrals are used for evaluation of (29) Oberhettinger[10] :

$$\int_0^\infty \sin\left(\sqrt{2k_2' uz}\right) e^{-ux}\, du = \frac{\sqrt{2k_2'\,\pi}}{2x^{3/2}} \exp\left(-\frac{k_2' z^2}{2x}\right) \qquad \ldots(33)$$

$$\int_0^\infty \sqrt{u}\cos\left(\sqrt{2k_2' uz}\right) e^{-ux}\, du = \frac{\sqrt{\pi}\,(x - k_2' z^2)}{2x^{5/2}} \exp\left(-\frac{k_2' z^2}{2x}\right). \qquad \ldots(34)$$

Further, when $x >> z$, then

$$r = (x^2 + z^2)^{1/2} = x\,(1 + z^2/x^2)^{1/2} = x + z^2/2x. \qquad \ldots(35)$$

From (39), it is obtained that

$$\psi_1(x, z) = \frac{\sqrt{\pi}\, e^{-k_2' r}}{2\sqrt{r}}\left[M(0)\,\frac{\sqrt{2k_2'}\,\sin\beta}{\cos^{3/2}\beta}\right.$$

$$\left. + N(0)\,\frac{\cos\beta - k_2'\, r\sin^2\beta}{r\cos^{5/2}\beta}\right] \qquad \ldots(36)$$

where $x = r\cos\beta$, $z = r\sin\beta$. Thus the scattered waves in (36) are transverse waves behaving as cylindrical waves. Their amplitude is of the form $\exp(-k_2' r)/\sqrt{r}$ which decays exponentially as the distance r from the scatterer at (0,0) increases. In the free surface ($z=0$), it can be seen that (33) and (34) behave as $O(x^{-3/2})$. Close to the scatterer as $r \to 0$, the amplitude of the scattered wave behaves as $O(r^{-3/2})$.

CONCLUSIONS

It is interesting to note that the scattered waves are transverse and not compressional. For the far-field when $x >> z$, the amplitude of the scattered wave behaves as exp $(-k_2' \, r)/\sqrt{r}$, $r = (x^2 + z^2)^{1/2}$, r being the distance from the scatterer at (0,0). The wave is a cylindrical wave. It dies out exponentially as it moves away from the scatterer. For the near-field as $r \rightarrow 0$, the scattered wave is of the form $O\,(r^{-3/2})$. It is dominant near the scatterer. The amplitude of the scattered wave is plotted versus the wave number (fig. 3). For Poisson's solids, when $\alpha = \sqrt{3}\ \beta, \theta = \pi/3$, the amplitude in the free surface ($z = 0$) of the quarter-space falls off sharply as the product of the wave number and the distance increases slowly. The amplitud has the value 36.60 when $k_2\,x$ = .1 and falls to the value .025 when $k_2\,x$ = 1.8. There is no scattered wave when $\theta = \pi/2$, i.e., when the wave is incident on the free surface parallel to the rigid boundary. The waves in (24-25) (both transverse and compressional) reflected from the boundaries of the medium satisfy the boundary conditions. The scattered waves together the waves contributed due to the branch cut at $p = k'$ also satisfy the boundary conditions.

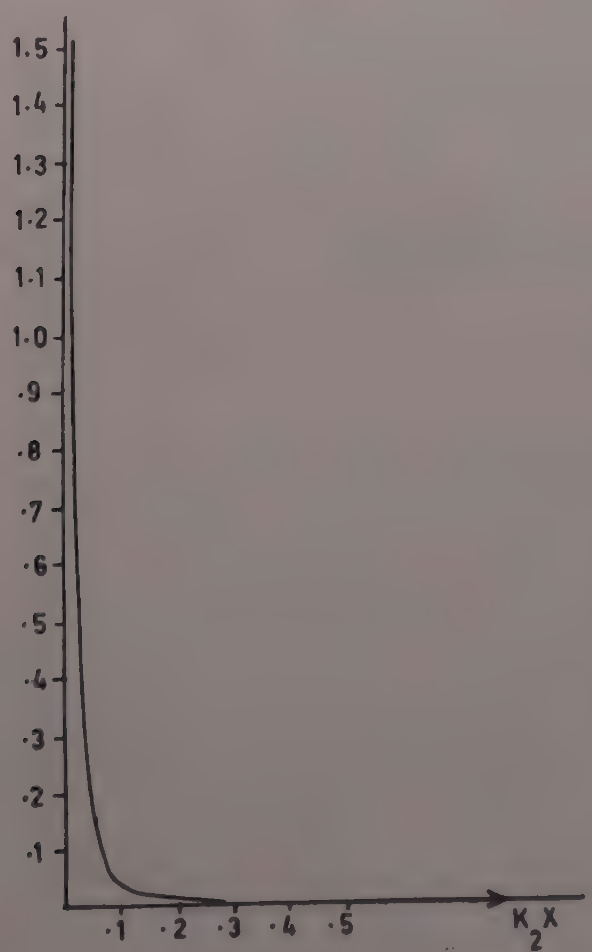

FIG. 3. Amplitude of the scattered wave.

The residues at the poles $p = \pm p_n$ contribute

$$\pm \frac{8\mu\, p_n\, k \cos\theta\, (2p_n^2 - k'^2)\, (Y_n - ik \sin\theta)}{(p_n^2 - k^2 \cos^2\theta)\, F'(\pm p_n)}\, e^{-\delta_n z}\, e^{\mp i p_n x}$$

where $Y_n = (p_n^2 - k^2)^{1/2}$, $\delta_n = (p_n^2 - k'^2)^{1/2}$. These are the Rayleigh waves as SV-type waves and P-type waves. These are surface waves. Their amplitude does not depend upon the wave number and is a constant multiple of the amplitude of the incident wave.

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ON TEMPERATURE-RATE DEPENDENT THERMOELASTIC LONGITUDINAL VIBRATIONS OF AN INFINITE CIRCULAR CYLINDER

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Green and Lindsay[1] have given the linearized temperature-rate dependent thermoelasticity theory. Using this theory the problem of thermoelastic longitudinal vibrations of an infinite circular cylinder has been solved and the results have been obtained in terms of potential functions. It is interesting to note that due to the temperature rate dependent theory the amplitude of both the elastic and thermal waves are higher than that of conventional theory. We further observe that if we put $\gamma = \gamma^* = 0$ in the results of this paper we arive at the results of the conventional theory of thermoelasticity.

1. INTRODUCTION

The thermoelasticity theory which includes the temperature-rate among the constitutive variables has aroused considerable interest in recent years. This theory which was developed by Green and Lindsay[1] and by Suhubi[2] is a generalisation of the conventional coupled thermoelasticity theory[3] and predicts a finite speed for the propagation of thermal signals. Several problems revealing interesting phenomena characterizing this theory have been considered earlier[4-6]. Because of the experimental evidence available in favour of the finiteness of heat propagation speed[7], these studies are of practical relevance too.

The problem of thermoelastic vibrations of a circular cylinder was discussed by Chadwick[8] and the solutions have been obtained in terms of potential functions. In this paper an attempt has been made to study the problem of thermoelastic longitudinal vibration of an infinite circular cylinder in the context of linearised temperature—rate dependent thermoelasticity theory and the solutions have been obtained in terms of the potential functions. It is interesting to note that due to the temperature rate dependent theory the amplitude of both the elastic and thermal waves are higher than that of conventional theory. We further observe that if we put $\gamma = \gamma^* = 0$ in the results of this paper we arrive at the results of the conventional theory of thermoelasticity discussed by Chadwick[8].

2. BASIC EQUATIONS

In the context of the linearised temperature—rate dependent thermoelastic theory of Green and Lindsay[1], the equations governing the displacement vector $\bar{U}$,

the stress tensor σ_{ij} and the temperature deviation θ, in a homogeneous and isotropic solid, in tensor notation are given as follows.

$$\sigma_{ij} = \lambda \, \delta_{ij} \, u_{k,k} + \mu \, (u_{i,j} + u_{j,i}) - \frac{\alpha}{\chi_T} \, (\theta + \gamma \, \dot{\theta}) \, \delta_{ij} \qquad \text{...(2.1)}$$

$$\rho \, \ddot{\bar{U}} = \mu \, \nabla^2 \, \bar{U} + (\lambda + \mu) \text{ grad div } \bar{U} - \frac{\alpha}{\chi_T} \text{ grad } (\theta + \gamma \dot{\theta}) \qquad \text{...(2.2)}$$

$$k \, \nabla^2 \, \theta = \rho c_\epsilon \, (\dot{\theta} + \gamma^* \, \ddot{\theta}) + \frac{\alpha \theta_0}{\chi_T} \dot{u}_{k,k}. \qquad \text{...(2.3)}$$

Here ρ is the mass density and λ and μ are the isothermal Lame's constants given by

$$\lambda = \frac{E_T \, \nu_T}{(1 + \nu_T) \, (1 - 2\nu_T)}, \; \mu = \frac{E_T}{2 \, (1 + \nu_T)} \qquad \text{...(2.4)}$$

in which E_T and ν_T are respectively isothermal Young's modulus and isothermal Poisson's ratio and

$$\alpha = \left(\frac{\partial \, u_{k,k}}{\partial \theta} \right)_\sigma \qquad \text{...(2.5)}$$

is the coefficient of volume expansion.

The isothermal compressibility is given by

$$\chi_T = 3 \left(\frac{\partial \, u_{k,k}}{\partial \, \sigma_{kk}} \right)_\theta = 3 \left(\frac{1 - 2\nu_T}{E_T} \right) \qquad \text{...(2.6)}$$

from which we see that E_T, ν_T as their suffixes indicate, are the isothermal elastic constants. Further C_ϵ is the specific heat at constant strain, k is the thermal conductivity, θ_0 is the initial uniform temperature and γ, γ^* are thermal constants characteristic of the theory. A superposed dot denotes partial differentiation with respect to time t. It is assumed that the body forces and heat sources are absent. Expressing the displacement vector $\bar{U}$ as the sum of irrotational and solenoidal components

$$\bar{U} = \bar{\nabla} \, \phi + \text{curl } \bar{A}. \qquad \text{.. (2.7)}$$

Equations (2.2) (2.3) are found to be equivalent to the set

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{E_T \, (1 - \nu_T)}{(1 + \nu_T) \, (1 - 2\nu_T)} \, \nabla^2 \, \phi - \frac{E_T \, \alpha}{3\rho \, (1 - 2\nu_T)} \, (\theta + \gamma \, \dot{\theta})$$

$$k \, \nabla^2 \, \theta = \rho C_\epsilon \, (\dot{\theta} + \gamma^* \, \ddot{\theta}) + \frac{E_T \, \theta_0 \, \alpha}{3 \, (1 - 2\nu_T)} \frac{\partial}{\partial t} \nabla^2 \, \phi \qquad \text{...(2.9)}$$

$$\frac{\partial^2 \, \bar{A}}{\partial t^2} = \frac{E_T}{2\rho \, (1 + \nu_T)} \, \nabla^2 \, \bar{A}. \qquad \text{...(2.10)}$$

From equation (2.8) we observe that the velocity of longitudinal elastic waves in a medium with zero coefficient of expansion is given by

$$V_T = \left[\frac{E_T\,(1 - \nu_T)}{(1 + \nu_T)\,(1 - 2\nu_T)} \right]^{1/2}. \qquad \ldots(2.11)$$

We shall refer to V_T as the isothermal velocity. Further the velocity of transverse elastic waves[8] is given by

$$V_s = \left[\frac{E_T}{2\,\rho(1 + \nu_T)} \right]^{1/2}. \qquad \ldots(2.12)$$

3. Thermoelastic Vibrations of a Circular Cylinder

We denote by (r, θ, z) cylindrical polar coordinates referred to the axis of the cylinder and make use of potential functions. In longitudinal disturbances the tangential component of displacement vanishes identically and since, in addition, field quantities do not depend upon the angular coordinates no confusion will arise from the use of θ to denote the temperature perturbation. The vector potential $\bar{A}$ is of the form $(0, \psi, 0)$ and the displacement components are given in terms of the scalar functions $\phi\,(r, z, t)$, $\psi\,(r, z, t)$ by

$$u_r = \frac{\partial \phi}{\partial r} - \frac{\partial \psi}{\partial z},\; u_\theta = 0,\; u_z = \frac{\partial \phi}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\,(r\,\psi). \qquad \ldots(3.1)$$

The thermoelastic equations (2.8) – (2.10) now take the form

$$\frac{\partial^2 \phi}{\partial t^2} = V_T^2 \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} - \frac{\alpha}{\rho \chi_T}\,(\theta + \gamma\dot{\theta}) \qquad \ldots(3.2)$$

$$\rho C_\varepsilon\,(\dot{\theta} + \gamma^*\,\ddot{\theta}) + \frac{\alpha \theta_0}{\chi_T}\frac{\partial}{\partial t}\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} \right)$$

$$= k\left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} + \frac{\partial^2 \theta}{\partial z^2} \right) \qquad \ldots(3.3)$$

$$\frac{\partial^2 \psi}{\partial t^2} = V_s^2\,\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\left(\frac{\partial \psi}{\partial r} - \frac{\psi}{r^2} + \frac{\partial^2 \psi}{\partial z^2} \right). \qquad \ldots(3.4)$$

Inserting into equations (3.2) – (3.4) the solution

$$[\theta, \phi, \psi] = [\hat{\theta}\,(r), \hat{\phi}\,(r), \hat{\psi}\,(r)]\; e^{i(\eta z - \omega t)}. \qquad \ldots(3.5)$$

In these expressions η and ω are, in general, complex quantities so that the wave length is $2\pi/\mathrm{Re}\,\eta$ and the period $2\pi/\mathrm{Re}\,\omega$.

$$\frac{d^2 \hat{\phi}}{dr^2} + \frac{1}{r}\frac{d\hat{\phi}}{dr} + \left(\frac{\omega^2}{V_T^2} - \eta^2 \right)\hat{\phi} = \frac{\alpha\,(1 - i\gamma\omega)\hat{\theta}}{\rho \chi_T\, V_T^2} \qquad \ldots(3.6)$$

$$\frac{d^2 \hat{\theta}}{dr^2} + \frac{1}{r}\frac{d\hat{\theta}}{dr} + \left[\frac{i\omega\rho C_\varepsilon}{k}(1 - i\,\gamma^*\omega) - \eta^2 \right]\hat{\theta}$$

(equation continued on p. 398)

$$= - \frac{i\omega\alpha\theta_0}{k\chi_T} \left(\frac{d^2 \hat{\theta}}{dr^2} + \frac{1}{r} \frac{d\hat{\phi}}{dr} - \eta^2 \hat{\phi} \right) \qquad \text{...(3.7)}$$

$$\frac{d^2 \hat{\psi}}{dr^2} + \frac{1}{r} \frac{d\hat{\psi}}{dr} + \left[\left(\frac{\omega^2}{V_s^2} - \eta^2 \right) - \frac{1}{r^2} \right] \hat{\psi} = 0. \qquad \text{...(3.8)}$$

Solving these equations, we get

$$\phi = A J_0 (r \sqrt{\xi_1^2 - \eta^2}) + B J_0 (r \sqrt{\xi_2^2 - \eta^2}) \exp \alpha(i(\eta z - \omega t)) \qquad \text{...(3.9)}$$

$$\theta = \frac{\rho \chi_T V_T^2}{\alpha (1 - i\gamma\omega)} \left[A \left(\frac{\omega^2}{V_T^2} - \xi_1^2 \right) J_0 \left(r \sqrt{\xi_1^2 - \eta^2} \right) \right.$$

$$\left. + B \left(\frac{\omega^2}{V_T^2} - \xi_2^2 \right) J_0 \left(r \sqrt{\xi_2^2 - \eta^2} \right) \right] \exp (i(\eta z - \omega t)) \qquad \text{...(3.10)}$$

$$\psi = C J_1 (r \sqrt{\xi_3^2 - \eta^2}) \exp (i (\eta z - \omega t)) \qquad \text{...(3.11)}$$

where ξ_1^2, ξ_2^2 are the roots of

$$\xi^4 - \left[\frac{\omega^2}{V_T^2} + \frac{i\omega\rho C_\varepsilon}{k} \left\{ (1 + \epsilon) - i\omega (\epsilon\gamma + \gamma^*) \right\} \right] \xi^2$$

$$+ \frac{i\omega^3 \rho C_\varepsilon}{k V_T^2} (1 - i\omega\gamma^*) = 0. \qquad \text{...(3.12)}$$

The equation corresponding to (3.12) from Chadwick[8] is

$$\xi^4 - \left[\frac{\omega^2}{V_T^2} + \frac{i\omega\rho C_\varepsilon}{k} (1 + \epsilon) \right] \xi^2$$

$$+ \frac{i\omega^3 \rho C_\varepsilon}{k V_T^2} = 0 \qquad \text{...(3.12A)}$$

$$\epsilon = \frac{\alpha^2 \theta_0}{\chi_T^2 C_\varepsilon \rho^2 V_T^2} \qquad \text{...(3.13)}$$

and

$$\xi_3^2 = \frac{\omega^2}{V_s^2}. \qquad \text{...(3.14)}$$

The Bessel function Y_0, Y_1 of the second kind are excluded from the solution by the requirement that u_r, u_z, θ shall be finite at the axis.

We find from (3.12) and (3.12A) that due to the introduction of the temperature rate theory the values of $\xi_1^2 + \xi_2^2$ and $\xi_1^2\ \xi_2^2$ are reduced by amounts of $i\omega\,(\epsilon\gamma + \gamma^*)$ and $i\omega\gamma^*$ respectively. This implies a reduction in the values of ξ_1^2 and ξ_2^2 which by equations (3.9) and (3.10) cause increase in the amplitude of both elastic and thermal waves.

The constants A, B, C are found by substituting (3.9) — (3.11) into boundary conditions which, since the surface of the cylinder is free from mechanical and thermal constraints, take the form

$$\sigma_{rr} = 0,\ \sigma_{rz} = 0 \text{ at } r = a. \qquad \ldots(3.15)$$

Equations (2.1) and (3.15) give

$$\left.\begin{aligned} \sigma_{rr} &= \rho V_T^2 \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} - 2\rho V_s^2 \left(\frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r \partial z} \right) - \frac{\alpha}{\chi_T}(\theta + \gamma\dot{\theta}) = 0 \right. \\ \sigma_{rz} &= \rho V_s^2 \left(2\frac{\partial^2 \phi}{\partial r\, \partial z} + \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} - \frac{\psi}{r^2} - \frac{\partial^2 \psi}{\partial z^2} \right) = 0. \\ &\text{Thermal boundary condition is} \\ &\frac{\partial \theta}{\partial r} + h\,\theta = 0 \end{aligned}\right\} \text{ at } r = a \qquad \ldots(3.16)$$

where h is the heat transfer coefficient of the boundary $r = a$.

The three homogeneous linear equations connecting A, B, C are found to be

$$A\left[\left(2 - \frac{\omega^2}{V_s^2\ \eta^2}\right) J_0(a\eta H_1) + \frac{2H_1}{a\eta}\ J_1(a\eta\, H_1)\right]$$

$$+ B\left[\left(2 - \frac{\omega^2}{V_s^2\ \eta^2}\right) J_0(a\,\eta\, H_2) + \frac{2H_2}{a\eta}\ J_1(a\,\eta\, H_2)\right]$$

$$- 2iCH_3\left[J_0(a\,\eta\, H_3) - \frac{1}{a\eta H_3}\ J_1(a\eta H_3)\right] = 0 \quad \ldots(3.17)$$

$$2i\,[AH_1\, J_1(a\eta H_1) + BH_2\, J_1(a\eta H_2) - C\left(2 - \frac{\omega^2}{V_s^2\ \eta^2}\right) J_1(a\eta H_3) = 0$$

$$\ldots(3.18)$$

$$A\left[\frac{h}{\eta}J_0(a\eta H_1)-H_1J_1(a\eta H_1)\right]\left[\left(\frac{\omega^2}{V_T^2\eta^2}-1-H_1^2\right)\right]$$

$$+B\left[\frac{h}{\eta}J_0(a\eta H_2)-H_2J_1(a\eta H_2)\right]\left(\frac{\omega^2}{V_T^2\eta^2}-1-H_2^2\right)=0 \qquad \ldots(3.19)$$

where

$$H_i^2=\frac{\xi_i^2}{\eta^2}-1, \qquad i=1,2,3. \qquad \ldots(3.20)$$

Eliminating A, B, C between equations (3.17) — (3.19) we obtain

$$\left(2-\frac{\omega^2}{V_s^2\eta^2}\right)^2 J_1(a\eta H_3)\left[H_1\left(H_1^2+1-\frac{\omega^2}{V_T^2\eta^2}\right)J_0(a\eta H_2)J_1(a\eta H_1)\right.$$

$$\left.-H_2\left(H_2^2+1-\frac{\omega^2}{V_T^2\eta^2}\right)J_0(a\eta H_1)J_1(a\eta H_2)\right]$$

$$+\frac{2}{a\eta}H_1H_2\left(H_1^2-H_2^2\right)J_1(a\eta H_1)J_1(a\eta H_2)$$

$$\times\left\{2aH_3J_0(a\eta H_3)-\frac{\omega^2}{V_s^2\eta^2}J_1(a\eta H_3)\right\}$$

$$=\frac{h}{\eta}\left[\left(2-\frac{\omega^2}{V_s^2\eta^2}\right)^2\left(H_1^2-H_2^2\right)J_0(a\eta H_1)\right.$$

$$\times J_0(a\eta H_2)J_1(a\eta H_3)-\frac{2}{a\eta}\left\{2a\eta H_3J_0(a\eta H_3)\right.$$

$$\left.-\frac{\omega^2}{V_s^2\eta^2}J_1(a\eta H_3)\right\}\times H_1\left(H_2^2+1-\frac{\omega^2}{V_T^2\eta^2}\right)$$

$$\times J_0(a\eta H_2)J_1(a\eta H_1)-H_2\left(H_1^2+1-\frac{\omega^2}{V_T^2\eta^2}\right)$$

$$\left.\times J_0(a\eta H_1)J_1(a\eta H_2)\Big\}\right]. \qquad \ldots(3.21)$$

Equations (3.12), (3.14), (3.20) and (3.21) together determine the wave number the wave number η as a function of the frequency ω.

We observe that the form of equation (3.21) is the same as the frequency equation obtained by Chadwick[8]. The difference lies only in the values of H_1 and H_2 which in the present case depend on γ and γ^* also and if we put $\gamma = \gamma^* = 0$ the two equations become identical and the results tally.

If we replace V_T by V_P and put

$$H_1 = \sqrt{\frac{V^2}{V_p^2} - 1}, \qquad H_2 = \infty \qquad \text{...(3.22)}$$

where

$$V = \frac{\omega}{\eta}. \qquad \text{...(3.23)}$$

Equation (3.21) reduces to

$$\left(2 - \frac{V^2}{V_s^2}\right)^2 J_0\left(a\eta\sqrt{\frac{V^2}{V_p^2} - 1}\right) J_1\left(a\eta\sqrt{\frac{V^2}{V_s^2} - 1}\right)$$

$$+ \left(\frac{2}{a\eta}\sqrt{\frac{V^2}{V_p^2} - 1}\right) J_1\left(a\eta\sqrt{\frac{V^2}{V_p^2} - 1}\right)$$

$$\times \left[\left(2a\eta\sqrt{\frac{V^2}{V_s^2} - 1}\right) J_0\left(a\eta\sqrt{\frac{V^2}{V_s^2} - 1}\right)\right.$$

$$\left. - \frac{V^2}{V_s^2} J_1\left(a\eta\sqrt{\frac{V^2}{V_s^2} - 1}\right)\right] = 0. \qquad \text{...(3.24)}$$

Equation (3.24) is the final result of the Pochammer Chree analysis of the longitudinal modes of Vibrations of a circular cylinder.

To find the order of magnitude of the discrepancy between equation (3.24) and the exact frequency relation (3.21) we expand the roots of equation (3.12) in powers of the reduced frequency χ^1. From equation (3.20) we obtain

$$H_1 = \sqrt{\frac{V^2}{V_p^2} - 1}\, [1 + O(\chi)] \qquad \text{...(3.25)}$$

$$H_2 = \frac{1+i}{\sqrt{2\chi}} \frac{V}{V_p} (1 + \epsilon) [1 + O(\chi)] \qquad \text{...(3.26)}$$

using the relation $V_p = V_T \sqrt{1 + \epsilon}$ ...(3.27)

(1) $\chi = \dfrac{\omega}{\omega^*}$ where $\omega = \dfrac{\rho C_\epsilon V_T^2}{k}$ is characteristic frequency of the solid[8].

It follows that

$$J_0\,(a\eta_i H_1) = J_0\left(a\eta\sqrt{\frac{V^2}{V_p^2}-1}\right) + O\,(\chi)\ \text{as}\ \chi \to 0$$

$$J_0\,(a\eta_i H_2) = (\tfrac{1}{2}\chi)^{1/2}\left\{\frac{V_p}{\pi a\eta_i V\,(1+\epsilon)}\right\}^{\frac{1}{2}}$$

$$\times\ \exp\left\{\frac{a\eta}{\sqrt{2\chi}}\frac{V}{V_p}(1+\epsilon)\,(1-i)+\frac{1}{8}\pi\,i\right\}, \qquad \ldots(3.28)$$

with similar results for the first order Bessel functions $J_1\,(a\eta_i H_1)$ $J_1\,(a\eta_i H_2)$. Entering these expressions into equation (3.21) it is found that each sider educes to multiple of $L + O(\chi)$ where L is the left hand side of equation (3.24). Thus in the limit $\chi \to 0$. V becomes independent of χ and h and we recover the classical frequency relation (3.24). Since this equation has been shown to differ from the thermoelastic frequency relation (3.21) by terms of order χ, the error sincurred by using the Pochammer—Chree analysis in the interpretation of experimental results are generally quite insignificant.

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LINEAR STABILITY AND THE RESONANCE FOR THE TRIANGULAR LIBRATION POINTS FOR THE DOUBLY PHOTOGRAVITATIONAL ELLIPTIC RESTRICTED PROBLEM OF THREE BODIES

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In the present paper we have found the range of values of μ and e for the linear stability of the triangular points for the doubly photo-gravitational elliptic restricted problem of three bodies and it has been shown that some resonances of the third and the fourth order exist which will need special investigation for the determination of complete stability of the lihration points under our reference.

1. INTRODUCTION

Simmons *et al.*[14], studied the existence and linear stability of the libration points for the photo-gravitational circular restricted problem of three bodies. They found the ranges of the linear stability as well for all the various libration points. Manju and Choudhry[9] studied the non-linear stability for resonance as well as for non-resonance cases for the isosceles triangular libration points when only one of the gravitating bodies was taken to be radiating and the reduction factor very small equal to that for the sum. The authors[5] studied the non-linear stability of the triangular libration points for non-resonance case for the doubly photo-gravitational restricted problem of three bodies where both of the gravitating bodies were assumed to be radiating and similar to the assumption of Simmons and others in the mentioned paper we took the reduction factors to vary from $-\infty$ to 1. In another paper[6] we have studied the non-linear stability of the same libration points in the presence of the third and the fourth order resonances.

In the present paper we have studied the linear stability and its range for the various admissible values of the eccentricity of the elliptic orbit. We have also shown that resonances of the third and the fourth order will exist under our range of linear stability. So for the study of non-linear stability we shall have to take into considera-

tion the resonance as well as non-resonance cases. Our present paper may be claimed to be a generalization over our mentioned papers in the sense that we consider here elliptic restricted problem and over Markeev's work[10] in the sense that we have here assumed both of the gravitating bodies to be radiating as well as where Markeev has taken them only gravitating. We have not investigated the existence of the various libration points in detail. However, the existence of three collinear and two triangular libration points can be shown. The present paper studies the linear stability of the triangular libration points only. A similar study for the collinear libration points will be interesting.

The present problem which was first taken up by Radziavskii[13] was studied in detail by Simmons *et al.*[14]. They remained restricted to the circular case and contented themselves with the linear stability only. Realising its far-reaching consequences we took up the investigation of the problem in full detail. The results are of immediate importance in stellar dynamics, but it is not less important for the solar system. Regarding the two reduction factors to be arbitrary parameters, the problem reduces to one where the gravitational forces may be taken to vary from the classical case ($\alpha = \beta = 1$) to the trivial case ($\alpha = \beta = 0$). Negative values for α and β can be available only for stellar system. Thus we find that the results are equally applicable to the solar system as well as to the stellar system. The circular restricted problem is only a first approximation of the problem. The natural problem is the elliptic restricted problem. To us it appears that the solution will be of far-reaching consequences applicable to solar as well as to stellar system.

2. Coordinates of the Triangular Libration Points

Let us refer our coordinates to Nechvil's coordinate system $(C, \xi\, \eta\, \zeta)$ (Duboshin[3]). We shall choose the sum of the two finite masses for the unit mass, the unit of time so that the constant of gravitation $k^2 = 1$ and for the unit length, the parameter p of the elliptic orbit. Then the equations of motion may be written as

$$\left.\begin{aligned} \frac{d\xi}{dv} &= \frac{\partial H}{\partial p_\xi}, & \frac{d\eta}{dv} &= \frac{\partial H}{\partial p_\eta}, & \frac{d\zeta}{dv} &= \frac{\partial H}{\partial p_\zeta} \\ \frac{dp_\xi}{dv} &= -\frac{\partial H}{\partial \xi}, & \frac{dp_\eta}{dv} &= -\frac{dH}{\partial \eta}, & \frac{dp_\zeta}{\partial \eta} &= -\frac{dH}{\partial \zeta} \end{aligned}\right\} \qquad \ldots(1)$$

where

$$H = \frac{1}{2}\left(p_\xi^2 + p_\eta^2 + p_\zeta^2\right) + p_\xi \eta - p_\eta \xi$$

$$+ \frac{e \cos v}{2\,(1 + e \cos v)}\,(\xi^2 + \eta^2 + \zeta^2) - \frac{W}{1 + e \cos v}$$

$$W = \alpha\,(1 - \mu)/r_1 + \beta\,\mu/r_2$$

$$r_1 = \sqrt{(\xi + \mu)^2 + \eta^2 + \zeta^2}\,, \quad r_2 = \sqrt{(\xi + \mu - 1)^2 + \eta^2 + \zeta^2}$$

μ, $1-\mu$ = the masses of the finite bodies ($0 < \mu < \frac{1}{2}$),

α, β = the reduction factors due to the radiation pressure exerted by the two finite bodies,

$(-\mu, 0, 0)$ = the coordinates of the larger finite body,

$(1-\mu, 0, 0)$ = the coordinates of the smaller finite body.

For the coordinates of the libration points, we consider the equations

$$\frac{\partial H}{\partial p_\xi} = \frac{\partial H}{\partial p_\eta} = \frac{\partial H}{\partial p_\zeta} = \frac{\partial H}{\partial \xi} = \frac{\partial H}{\partial \eta} = \frac{\partial H}{\partial \zeta} = 0 \qquad \text{...(2)}$$

whence we find that $r_1 = \alpha^{1/3}$, $r_2 = \beta^{1/3}$, is a solution of the equations. If L_4 and L_5 be the corresponding libration points, then clearly they are seen to form triangels with the finite masses for the other vertices. For convenience, if we put $\alpha = \delta_1^3$ and and $\beta = \delta_2^3$ and $(\xi_{Li}, \eta_{Li}, \zeta_{Li}, p_{\xi_{Li}}, p_{\eta_{Li}}, p_{\zeta_{Li}})$ $(i = 4, 5)$ be taken for the coordinates of L_i $(i = 4, 5)$, then

$$\xi_{L_\xi} = \xi_{L_\xi} = \frac{\delta_1^2 + 1 - \delta_2^2}{2} - \mu, \; p_{\xi_{L_\zeta}} = -p_{\xi_{L_\zeta}} = -\delta_1 \delta_2 \sqrt{b}$$

$$\eta_{L_4} = \eta_{L_5} = \delta_1 \delta_2 \sqrt{b}, \; p_{\eta_{L_4}} = p_{\eta_{L_5}} = \frac{\delta_1^2 + 1 - \delta_2^2}{2} - \mu$$

$$\zeta_{L_4} = \zeta_{L_5} = 0, \; p_{\zeta_{L_4}} = p_{\zeta_{L_5}} = 0$$

$$b = 1 - \left(\delta_1^2 + \delta_2^2 - 1\right)^2 \Big/ 4\,\delta_1^2 + \delta_2^2 .$$

Different from the classical case[3] here the triangles will be ordinary triangles whose sides will be of lengths δ_1, δ_2 and 1.

3. Characteristic Roots and the First Order Stability

We shall restrict our study to the planar case alone. Since L_5 is symmetrical to L_4, the nature of motion near L_5 will be the same as near L_4, so we shall consider the motion near L_4 alone. Taking (q_i, p_i) $(i = 1, 2)$ for the variations in the coordinates of L_4, the variational equations may be written as

$$\frac{dq_i}{dv} = \frac{\partial H}{\partial p_i}, \; \frac{dp_i}{dv} = -\frac{\partial H}{\partial q_i} \; (i = 1, 2) \qquad \text{...(3)}$$

where

$$H = \sum_{k=0}^{\infty} H_k, \; H_0 = \text{constant}, \; H_1 = 0$$

$$H_2 = \frac{1}{2}\left(p_1^2 + p_2^2\right) + p_1 q_2 - q_1 p_2 + \frac{e \cos v}{2(1 + e\cos v)}\left(q_1^2 + q_2^2\right)$$

$$+ \frac{q_1^2}{2(1 + e\cos v)}\left[1 - \frac{3}{4}\left\{(1-\mu)\,\xi^2\,\delta_1^{-2} + \mu\,\eta^2\,\delta_2^{-2}\right\}\right]$$

$$- \frac{3\sqrt{b}\,q_1 q_2}{2(1 + e\cos v)}\left[(1-\mu)\,\xi\,\delta_2\,\delta_1^{-1} + \mu\,\eta\,\delta_1\,\delta_2^{-1}\right]$$

$$- \frac{q_2^2}{2(1 + e\cos v)}\left[3b\left\{(1-\mu)\,\delta_2^2 + \mu\,\delta_1^2\right\} - 1\right] \qquad \ldots(4)$$

$$H_3 = \frac{1}{1 + e\cos v}\left[\frac{q_1^3}{1b}\left\{(1-\mu)\,\xi\,(5\xi^2 - 12\delta_1^2)\,\delta_1^{-4}\right.\right.$$

$$\left. + \mu\,\eta\,(5\eta^2 - 12\delta_2^2)\,\delta_2^{-4}\right\} + \frac{3}{8}\,q_1^2 q_2\sqrt{b}\left\{(1-\mu)\,\delta_2\,\delta_1^{-3}\,(5\xi^2 - 4\delta_1^2)\right.$$

$$\left. + \mu\,\delta_1\,\delta_1^{-3}\,(5\eta^2 - 4\delta_2^2)\right\} + \frac{3}{4}\,q_1 q_2^2\left\{\xi\,\delta_1^{-2}(1-\mu)(5b\,\delta_2^2 - 1)\right.$$

$$\left. + \eta\,\delta_2^{-2}\,\mu\,(5b\,\delta_1^2 - 1)\right\} + \frac{1}{2}\,q_2^3\sqrt{b}\left\{(1-\mu)\,\delta_2\,\delta_1^{-1}(5\,\delta_2^2\,b - 3)\right.$$

$$\left.\left. + \mu\,\delta_1\,\delta_2^{-1}(5\,\delta_1^2\,b - 3)\right\}\right]. \qquad \ldots(5)$$

$$H_4 = \frac{1}{1 + e\cos v}\left[-\frac{1}{8}\,q_1^4\left\{(1-\mu)\,\delta_1^{-6}\left(3\,\delta_1^4 - \frac{15}{2}\xi^2\,\delta_1^2 + \frac{35}{16}\,\xi^4\right)\right.\right.$$

$$\left. + \mu\,\delta_2^{-6}\left(3\,\delta_2^4 - \frac{15}{2}\,\eta^2\,\delta_2^2 + \frac{35}{16}\,\eta^4\right)\right\}$$

$$+ \frac{5}{4}\,q_1^3 q_2\sqrt{b}\left\{(1-\mu)\,\delta_2\,\delta_1^{-3}\,\xi\left(3 - \frac{7}{4}\,\xi^2\,\delta_1^{-2}\right)\right.$$

$$\left. + \mu\,\delta_1\,\delta_2^{-3}\,\eta\left(3 - \frac{7}{4}\,\eta^2\,\delta_2^{-2}\right)\right.$$

$$+ \frac{5}{4}\,q_1 q_2^3\sqrt{b}\left\{(1-\mu)\,\xi\,\delta_2\,\delta_1^{-3}\left(3 - 7\,\delta_2^2\,b\right)\right.$$

$$\left. + \mu\,\eta\,\delta_1\,\delta_2^{-3}\left(3 - 7\,\delta_1^2\,b\right)\right\}$$

$$+ \frac{3}{4}\,q_1^2 q_2^2\left\{(1-\mu)\,\delta_1^{-2}\left(-1 + 5\,\delta_2^2\,b + \frac{5}{4}\,\xi^2\,\delta_1^{-2}\right.\right.$$

$$\left. - \frac{35}{4}\,\xi^2\,\delta_1^{-2}\,\delta_2^2\,b\right) + \mu\,\delta_2^{-2}\left(-1 + 5\,\delta_1^2 + \frac{5}{4}\,\eta^2\,\delta_2^{-2}\right.$$

$$\left.\left. - \frac{35}{4}\,\eta^2\,\delta_1^2\,\delta_2^{-2}\,b\right)\right\} - \frac{q_1^4}{8}\left\{(1-\mu)\,\delta_1^{-2}\left(3 - 30b\,\delta_2^2 + 35\,\delta_2^4\,b^2\right)\right.$$

(equation continued on p. 407)

$$+ \mu \, \delta_2^{-2} \left(3 - 30b \, \delta_1^2 + 35 \, \delta_1^4 b^2 \right) \Big\} \Big] . \qquad \ldots(6)$$

H_k = the sum of the terms of the kth degree homogeneous in the variables q_1, q_2, p_1 and p_2

$$\xi = \delta_1^2 + 1 - \delta_2^2 \quad \text{and} \quad \eta = \delta_1^2 - 1 - \delta_2^2 .$$

To find the characteristic roots we shall follow Bennet's method[1] where he expands λ, H_2 as well as the general solution of the first order variational equation in powers of e and also uses

$$(1 + e \cos v)^{-1} = 1 - e \cos v + e^2 \cos^2 v - e^3 \cos^3 v + \ldots\ldots$$
$$+ (-1)^r e^r \cos^r v + \ldots\ldots \qquad \ldots(1)$$

If the general solution be put as

$$y = y^{(0)} + y^{(1)} e + y^{(2)} e^2 + \ldots\ldots\ldots \qquad \ldots(8)$$

then equating the coefficients of different powers of e on the two sides of the variational equation, we get differential equations for $y^{(0)}$, $y^{(1)}$, $y^{(2)}$ and so on and they can be solved successively. Now putting these solutions in the differential equations for $y^{(0)}$, $y^{(1)}$, $y^{(2)}$ and so on, the coefficients $\lambda^{(0)}$, $\lambda^{(1)}$, $\lambda^{(2)}$ are calculated. which may be given as

$$\lambda^{(1)} = \pm \left[\tfrac{1}{2} \pm \tfrac{1}{2} \{1 - 36 \mu (1 - \mu) b\}^{1/2} \right] \qquad \ldots(9)$$

$$\lambda^{(1)} = 0 \quad \text{and} \quad \lambda^{(2)} = \frac{\alpha \{\lambda^{(0)}\}^2 + \beta}{\gamma \{\lambda^{(0)}\}^2 + \delta} \lambda^{(0)}$$

where α, β, γ and δ are constants depending on μ.

In order that the roots (9) may be purely imaginary, it will be necessary that

$$36 \mu (1 - \mu) b \leqslant 1 \Leftrightarrow \mu \leqslant 0.0285954 \qquad \ldots(10)$$

$\mu = 0.0285954\ldots\ldots$ corresponds to the resonance case with equal frequencies for $b = 1$ and $e = 0$. This limit coincides with the limit of stability found by Lanzano[7]. It may be noted for distinct δ_1 and δ_2 we shall have different ranges of stability given by the different values of μ for $e = 0$ shown by us in our earlier paper[6].

Lukyanov[8] uses Bennet's method as well as he adopts another method by expanding $1/(1 + e \cos v)$ in Fourier series given as

$$1/(1 + e \cos v) = \frac{1}{\sqrt{1 - e^2}} (1 + 2 \epsilon \cos v + 2 \epsilon^2 \cos 2v + \ldots\ldots)$$

where

$$\epsilon^r = \frac{1}{\pi}\int_0^{\pi} \cos rv \, d\, v/(1 + e\cos v)$$

and then proceeding similar to Bennet he calculates the coefficients $\bar{\lambda}^{(0)}, \bar{\lambda}^{(1)}, \bar{\lambda}^{(2)}$ and so on, of the charasteristic exponent $\bar{\lambda}$. By calculating the characteristic exponents by the two methods, he finally compares the ranges of the linear stability, obtained by the two methods and he shows that the ranges coincide with those found by Danby[4]. If $\bar{\lambda}^{(0)}$ be the characteristic root corresponding to the case $\epsilon = 0$, then

$$\bar{\lambda}^{(0)} = \pm\left[-\frac{1}{2}\left(4 - \frac{3}{\sqrt{1-e^2}}\right) \pm \frac{1}{2}\left\{\left(4 - \frac{3}{\sqrt{1-e^2}}\right)^2 - \frac{36\,\mu\,(1-\mu)\,b}{3-e^2}\right\}^{1/2}\right]^{1/2} \qquad \ldots(11)$$

whence we find that for purely imaginary value of $\bar{\lambda}^{(0)}$

$$e < 0.66144\ldots\ldots \text{ and } (4 - 3/\sqrt{1-e^2})^2 \geqslant 36\,\mu\,(1-\mu)\,b/(1-e^2). \qquad \ldots(12)$$

It gives

$$\mu \leqslant \tfrac{1}{2} - \sqrt{\tfrac{1}{4} - (25 - 16\,e^2 - 24\sqrt{1-e^2})\,/36\,b}. \qquad \ldots(13)$$

The equality corresponds to resonance cases with equal frequencies for different values of b. We shall not study these cases in further details. The values of μ denoted by $\mu\,(e)$ basing on the range of vales of b varying from 0 to 1 are given in Table I.

4. Normalization of the Hamiltonian Function H_2

Taking into view our subsequent studies we shall need the normalization of H_2 given by (4). For convenience we shall write H_2 in the form

$$H_2 = H_2^{(0)} + H_2^{(1)} \qquad \ldots(14)$$

where

$$H_2^{(0)} = \frac{1}{2}\left(p_1^2 + p_2^2\right) + p_1\,q_2 - q_1\,p_2 + \frac{1}{2}\,q_1^2\,(1-A) - \frac{1}{2}\,B\,q_1\,q_2 + \frac{1}{2}\,q_2^2\,(1-C) \qquad \ldots(14a)$$

$$H_2^{(1)} = (e\cos v/2\,(1 + e\cos v))\left(A\,q_1^2 + B\,q_1\,q_2 + C\,q_2^2\right) \qquad \ldots(14b)$$

TABLE I

The values of μ (e)

b	e	μ(e)	b	e	μ(e)	b	e	μ(e)	b	e	μ(e)
0.00	00—										
	0.7	imaginary	.3	.3	0.0659684	.6	00	0.0486645	.8	.5	0.00753561
0.05	00—07	,,	.3	.4	0.0429195		.1	0.046633		.6	0.00139082
.1	00	,,	.3	.6	0.203579		.2	0.0407785		.7	0.00071481
.1	.1	,,	.3	.6	0.0037175		.3	0.0318208	.9	00	0.0318805
.1	.2	0.376283	.3	.7	0.0019084		.4	0.0209788		.1	0.0305737
.1	.3	0.244755	.4	00	0.0750817		.5	0.0100732		.2	0.026795
1	.4	0.143956	.4	.1	0.0007185		.6	0.00185529		.3	0.00209989
.1	.5	0.0639155	.4	.2	0.0625911		.7	0.00095330		.4	0.0138852
.1	.6	0.011237	.4	.3	0.0485716	.7	00	0.0413961		.5	0.00669264
.1	.7	0.005747		.4	0.0318206		.1	0.03968		.6	0.00123609
.2	00	0 1667		.5	0.0151883		.2	0.03473		.7	0.000635
.2	.1	0 158496		.6	0.0278553		.3	0.02714	.1	00	0.0285954
.2	.2	0.135784		.7	0.00143063		.4	0.017918		.1	0.0274273
.2	.3	0.1030426	.5	00	0.0590414		.5	0.00862		.2	0.0240476
.2	.4	0.0659679		.1	0.0565477		.6	0.00159		.3	0.0188399
.2	.5	0.0308681		.2	0 0493768		.7	0.000817		.4	0.0124789
.2	.6	0.0058676		.3	0.0384482	.8	00	0.0360196		.5	0.0060192
.2	.7	0.0028654		.4	0.0252858		.1	0.0345365		.6	0.0011123
.3	00	0.1032539		.5	0.0121128		.2	0.0302519		.7	0.000571
.3	.1	0.0986483		.6	0.0022272		.3	0.0236663			
.3	.2	0.0853556		.7	0.00114428		.4	0.0156489			

$$\left.\begin{aligned} A &= \frac{3}{4}\left\{(1-\mu)\,\xi^2\,\delta_1^{-2} + \mu\,\eta^2\,\delta_2^{-2}\right\} \\ B &= \frac{3\sqrt{b}}{\delta_1\,\delta_2}\left\{\delta_2^2\,\xi\,(1-\mu) + \delta_1^2\,\eta\,\mu\right\} \\ C &= 3b\left\{(1-\mu)\,\xi_2^2 + \mu\,\delta_1^2\right\}. \end{aligned}\right\} \qquad \text{...(15)}$$

For normalisation we shall introduce the transformation

$$(q_1, q_2, p_1, p_2) = \left(q_1', q_2', p_1', p_2'\right) N \qquad \text{...(16)}$$

where N is the same as used in Manju and Choudhry[9].

Here

$$\omega_1^2 = -\left\{\lambda_{1,2}^{(0)}\right\}^2 \text{ and } \omega_2^2 = -\left\{\lambda_{3,4}^{(0)}\right\}^2$$

we shall call ω_1 and ω_2 as the frequencies and they are given by the equation

$$\omega^4 - \omega^2 + 9\mu(1 - \mu) b = 0. \qquad \ldots(17)$$

The transformation (16) reduces the Hamiltonian (14) to the form

$$H_2^1 = \frac{1}{2}\left(p_1^{12} + \omega_1^2 q_1^{12}\right) - \frac{1}{2}\left(p_2^{12} + \omega_2^2 q_2^{12}\right)$$

$$+ \frac{e \cos v}{1 + e \cos v} \sum_{\nu+\mu=2} a^1_{\nu\mu} q_1^{1\nu_1} q_2^{1\nu_2} p_1^{1\mu_1} p_2^{1\mu_2} \qquad \ldots(18)$$

where $\nu = (\nu_1, \nu_2)$, $\mu = (\mu_1, \mu_2)$ and ν_1, ν_2, μ_1 and μ_2 so vary that $\nu + \mu = \nu_1 + \nu_2 + \mu_1 + \mu_1 = 2$ and for simplicity we shall not write the ranges again and the coefficients $a'_{\nu\mu}$ are given as follows :

$$a'_{2000} = \frac{1}{2} a_1^2\left(A + B c_1 + C c_1^2\right),\ a'_{0200} = \frac{1}{2} a_2^2\left(A + B c_2 + C c_2^2\right)$$

$$a'_{0200} = \frac{1}{2} C a_1^2 b_1^2,\quad a'_{0002} = \frac{1}{2} C a_2^2 b_2^2$$

$$a'_{1100} = a_1 a_2 \left\{A + \frac{1}{2} B(c_1 + c_2) + C c_1 c_2\right\}$$

$$a'_{1010} = a_1^2 b_1 (\tfrac{1}{2} B + C c_1),\quad a'_{1001} = - a_1 a_2 b_2 (\tfrac{1}{2} B + C c_1)$$

$$a'_{0110} = a_1 a_2 b_1 (\tfrac{1}{2} B + C c_2),\quad a'_{0011} = - a_1 a_3 b_1 b_2 C$$

$$a'_{0101} = - a_2^2 b_2 (\tfrac{1}{2} B + C c_2).$$

Next we shall introduce the following transformations :

$$q'_i = \frac{1}{\sqrt{\omega_i}} \tilde{q}_i,\quad p'_i = \sqrt{\omega_i}\, \tilde{p}_i \quad (i = 1, 2) \qquad \ldots(19)$$

which transform $H_2^{'(0)}$ to $\tilde{H}_2^{(0)}$ given as

$$\tilde{H}_2^{(0)} = \tfrac{1}{2}\omega_1\left(\tilde{p}_1^2 + \tilde{q}_1^2\right) - \tfrac{1}{2}\omega_2\left(\tilde{p}_2^2 + \tilde{q}_2^2\right) \qquad \ldots(20)$$

and $H_2^{'(1)}$ to $\tilde{H}_2^{(1)}$ given as

$$H_2^{(1)} = \frac{e \cos v}{1 + e \cos v} \sum_{\nu+\mu=2} a_{\nu\mu} \tilde{q}_1^{\nu_1} \tilde{q}_2^{\nu_2} \tilde{p}_1^{\mu_1} \tilde{p}_2^{\mu_2} \qquad \ldots(21)$$

where $\tilde{a}_{\nu\mu}$ can be easily calculated. For the convenience of subsequent calculations we shall introduce the complex conjugate canonic variables given as

$$q_j' = \tilde{p}_j + i\,\tilde{q}_j, \quad p_j' = \tilde{p}_j - i\,\tilde{q}_j \quad (j = 1, 2). \qquad \ldots(22)$$

Consequently the Hamiltonian will be reduced to the form $H_2' = 2i\,\tilde{H}_2$

where

$$H_2' = i\,\omega_1\, q_1''\, p_2' - i\,\omega_2\, q_2'\, p_2' + 2i\,\frac{e\cos v}{1 + e\cos v} \sum_{\nu+\mu=2} a'_{\nu\mu}\, g_1^{'\nu_1}\, q_2^{'\nu_2}\, p_1^{'\mu_1}\, p_2^{'\mu_2} \qquad \ldots(23)$$

The coefficients in H_2' are such that $a''_{\nu\mu} = \tilde{a}'_{\mu\nu}$

where the bar sign denotes the complex conjugate quantity. Other coefficients are given as follows :

$$a'_{2000} = \tfrac{1}{4}\left(-\tilde{a}_{2000} + \tilde{a}_{0020} - i\,\tilde{a}_{1010}\right)$$

$$a'_{0200} = \tfrac{1}{4}\left(-\tilde{a}_{0200} + \tilde{a}_{0002} - i\,\tilde{a}_{0101}\right)$$

$$a'_{1100} = \tfrac{1}{4}\left(-\tilde{a}_{1100} + \tilde{a}_{0011} - i\,\tilde{a}_{1001} - i\,\tilde{a}_{0110}\right)$$

$$a''_{1001} = \tfrac{1}{4}\left(-\tilde{a}_{1100} + \tilde{a}_{0011} - i\,\tilde{a}_{1001} - i\,\tilde{a}_{0110}\right)$$

$$a'_{1010} = \tfrac{1}{2}\left(\tilde{a}_{2000} + \tilde{a}_{0020}\right), \quad \tilde{a}''_{0101} = \tfrac{1}{2}\left(\tilde{a}_{0200} + \tilde{a}_{0002}\right). \qquad \ldots(24)$$

Next we shall find the transformation

$$\left(q_j', p_j''\right) \to \left(q_j^{**}, p_j^{**}\right) \qquad \ldots(25)$$

reducing the Hamiltonian (23) to the normal form in complex conjugate variables given as

$$H_2^*\left(q_j^{**}, p_j^{**}\right) = i\,\lambda_1\, q_1^{**}\, p_1^{**} + i\,\lambda_2\, q_2^{**}\, p_2^{**}. \qquad \ldots(26)$$

Let this transformation be given by the generating function

$$q_1''\, p_1^{**} + p_2''\, q_2^{**} + S\left(q_1'', q_2', p_1^{**}, p_2^{**}, v\right)$$

where

$$S = \sum_{\nu+\mu=2} S_{\nu\mu}\, q_1''^{\nu_1}\, q_2''^{\nu_2}\, p_1^{**\mu_1}\, p_2^{**\mu_2}$$

and $s_{\nu\mu}$ are to be chosen 2π-periodic functions of v. The relation between the variables q_j'', p_j'' and q_j^{**}, p_j^{**} are given as

$$q_j^{**} = q_j'' + \frac{\partial s}{\partial p_j^{**}}, \quad p_j'' = p_j^{**} + \frac{\partial s}{\partial p_j''}$$

whence we the identity

$$H_2^{**}\left(q_j'' + \frac{\partial s}{\partial p_j^{**}}, p_j^{**}, v\right) - H_2''\left(q_j'', p_j^{**} + \frac{\partial s}{\partial p_j''}, v\right) = \frac{\partial s}{\partial v}.$$

On expanding and equating the coefficients of equal powers on the two sides, we shall get

$$H_2^{**}\left(q_j'', p_j'', v\right) + \sum_{j=1}^{2} \frac{\partial s}{\partial p_j^{**}} \frac{\partial H_2^{**}}{\partial q_j''} + \frac{1}{2}\left[\left(\frac{\partial s}{\partial p_1^{**}}\right)^2 \frac{\partial^2 H_2}{\partial q_1''^2}\right.$$

$$+ 2\frac{\partial s}{\partial p_1^{**}} \frac{\partial s}{\partial p_2^{**}} \frac{\partial^2 H_2^{**}}{\partial q_1'' \partial q_2''} + \left.\left(\frac{\partial s}{\partial p_2^{**}}\right)^2 \frac{\partial^2 H_2^{**}}{\partial q_2''^2}\right]$$

$$- H_2''\left(q_j'', p_j^{**}, v\right) - \sum_{j-j}^{2} \frac{\partial s}{\partial q_j''} \frac{\partial H_2''}{\partial p_j^{**}}$$

$$- \frac{1}{2}\left[\left(\frac{\partial s}{\partial q_1''}\right)^2 \frac{\partial^2 H_2''}{\partial p_1^{**2}} + 2\left(\frac{\partial s}{\partial q_1''}\right)\left(\frac{\partial s}{\partial q_2''}\right)\right.$$

$$\times \frac{\partial^2 H_2''}{\partial p_1^{**} \partial p_2^{**}} + \left.\left(\frac{\partial s}{\partial q_2''}\right)^2 \frac{\partial^2 H_2^{**}}{\partial p_2^{**2}}\right]$$

$$= \sum_{\nu+\mu=2} \frac{ds_{\nu\mu}}{dv}\, q_1''^{\nu_1}\, q_2''^{\nu_2}\, p_1^{**\mu_1}\, p_2^{**\mu_2} \qquad \ldots(27)$$

whence we get

$$i\,\lambda_1\,q_1''\,p_1^{**} + i\,\lambda_2\,q_2''\,p_2^{**} + i \sum_{\upsilon+\mu=2} (\mu_1\,\lambda_1 + \mu_2\,\lambda_2)\, s_{\nu\mu}\, q_1^{''\nu_1}\, q_2^{''\upsilon_2}$$

$$\times\, p_1^{**\mu_1}\, p_2^{**\mu_2} - i\,\omega_1\,q_1''\,p_1^{**} + i\,\omega_2\,q_2''\,p_2^{**}$$

$$-\,2i\,\frac{e\cos\nu}{1+e\cos\nu} \sum_{\upsilon+\mu=2} a''_{\nu\mu}\, q_1^{''\upsilon_1}\, q_2^{''\nu_2}\, p_1^{**\mu_1}\, p_2^{**\mu_2}$$

$$-\sum_{\upsilon+\mu=2} (\nu_1\,\omega_1 - \nu_2\,\omega_2)\, s_{\nu\mu}\, q_1^{''\upsilon_1}\, q_2^{''\upsilon_2}\, p_1^{**\mu_1}\, p_2^{**\mu_2}$$

$$= \sum_{\upsilon+\mu=2} \frac{ds_{\nu\mu}}{dv}\, q_1^{''\upsilon_1}\, q_2^{''\upsilon_2}\, p_1^{**\mu_1}\, p_2^{**\mu_2}\,. \qquad \ldots(28)$$

Restricting only upto the second order terms in e, we may write (28) as

$$i\,\lambda_1\,q_1''\,p_1^{**} + i\,\lambda_2\,q_2''\,p_2^{**} + i \sum (\mu_1\,\lambda_1 + \mu_2\,\lambda_2)\,[e\,s^{(1)} + e^2\,s^{(2)})$$

$$-\,i\,\omega_1\,q_1''\,p_1^{**} + i\,\omega_2\,q_2''\,p_2^{**} - 2i\left[e\cos\nu - \frac{e^2}{2}(1+\cos 2\nu)\right]$$

$$\times \sum a''_{\nu\mu}\, q_1^{''\nu_1}\, q_2^{''\upsilon_2}\, p_1^{**\mu_1}\, p_2^{**\mu 2}$$

$$-\,i\,\Sigma\,(\nu_1\,\omega_1 - \nu_2\,\omega_2)\,(e\,s^{(1)} + e^2\,s^{(2)})$$

$$= e\,\frac{ds^{(1)}}{dv} + e^2\,\frac{ds^{(2)}}{dv} \qquad \ldots(29)$$

where

$$s_{\nu\mu} = e \sum s^{(1)}_{\upsilon\mu} + e \sum s^{(2)}_{\nu\mu}\,.$$

Let us write

$$\left.\begin{aligned} \lambda_1 &= \lambda_1^{(0)} + e\,\lambda_1^{(1)} + e^2\,\lambda_1^{(2)} + \ldots \\ &\text{and} \\ \lambda_2 &= \lambda_2^{(0)} + e\,\lambda_2^{(1)} + e^2\,\lambda_2^{(2)} + \ldots \end{aligned}\right\} \qquad \ldots(30)$$

and equate the coefficients of the equal powers in e, then we shall have

$$\lambda_1^{(0)} = \omega_1,\ \lambda_2^{(0)} = \omega_2,\ i\,\lambda_1^{(1)} + i\,\lambda_2^{(1)} + i \sum\left(\mu_1\,\lambda_1^{(0)} + \mu_2\,\lambda_2^{(0)}\right) s^{(1)}$$

$$-\,2i\cos\nu\, a''_{\nu\mu} - i \sum (\nu_1\,\omega_1 - \nu_2\,\omega_2) = \frac{ds^{(1)}}{dv} \qquad \ldots(31)$$

whence we shall have the following relations.

$$
\left.\begin{aligned}
\frac{ds^{(1)}_{1010}}{dv} &= i\,\lambda^{(1)}_1 + i\,\lambda^{(0)}_1\, s^{(1)}_{1010} - 2i\cos v\; a''_{1010} - i\,\omega_1\, s^{(1)}_{1010}\\
\frac{ds^{(1)}_{0101}}{dv} &= i\,\lambda^{(1)}_2 + i\,\lambda^{(0)}_2\, s^{(1)}_{0101} - 2i\cos v\; a''_{0101} - i\,\omega_2\, s^{(1)}_{0101}\\
\frac{ds^{(1)}_{\nu\mu}}{ds} &+ i\,[(\nu_1-\mu_1)\,\omega_1 - (\nu_2-\mu_2)\,\omega_2]\; s^{(1)}_{\nu\mu} = -\,2i\cos v\; a''_{\nu\mu}
\end{aligned}\right\} \quad \ldots(32)
$$

On integration, we shall have

$$
s^{(1)}_{1010} = i\,\lambda^{(1)}_1\, v - 2i\sin v\; a''_{1010}
$$

$$
s^{(1)}_{0101} = -\,2i\sin v\; a''_{0101} + i\,\lambda^{(1)}_2\, v
$$

$$
s^{(1)}_{\nu\mu} = \frac{2i\,a''\nu\mu\;[\sin\nu + i\,\{(\nu_1-\mu_1)\,\omega_1 - (\nu_2-\mu_2)\,\omega_2\}\cos\nu]}{[(\nu_1-\mu_1)\,\omega_1 - (\nu_2-\mu_2)\,\omega_2]^2 - 1}. \quad \ldots(33)
$$

By virtue of periodicity of $s^{(1)}_{1010}$ and $s^{(1)}_{0101}$ it follows that $\lambda^{(1)}_1 = \lambda^{(1)}_2 = 0$ and the relations (33) completely determine S as a complex-valued function restricted to the first order terms in e alone.

Now let us find a real-valued transformation $(\tilde{q}_j, \tilde{p}_j) \to (q^*_j, p^*_j)$ reducing the Hamiltonian $\tilde{H}_2 = \tilde{H}^{(0)}_2 + \tilde{H}^{(1)}_2$ to the normal form given as

$$
H_2 = \tfrac{1}{2}\lambda_1\,(q^{*2}_1 + p^{*2}_1) + \tfrac{1}{2}\,\lambda_2\,(q^{*2}_2 + p^{*2}_2) \quad \ldots(34)
$$

Let this transformation be given by means of the generating function $\tilde{q}_1\, p^*_1 + \tilde{q}_2\, p^*_2 + K(\tilde{q}_j, p^*_j, v)$, where K is restricted to the order of e alone. From the transformation formula

$$
q^*_j = \tilde{q}_j + \frac{\partial K}{\partial p^*_j}, \quad \tilde{p}_j = p^*_j + \frac{\partial K}{\partial \tilde{q}^*_j}
$$

we may obtain by implicit function theorem and using the fact that all the variables are small

$$
\tilde{q}_j = q^*_j - \frac{\partial K}{\partial p^*_j}, \quad \tilde{p}_j = p^*_j + \frac{\partial K}{\partial q^*_j} \quad \ldots(35)
$$

where K^* will be obviously of the order of e.

By the formula (26) correct to the order of e we have

$$q_j^{*} = q_j^{**} - \frac{\partial S^{**}}{\partial p_j^{**}} \quad p_j^{*} = p_j^{**} + \frac{\partial S^{**}}{\partial q_j^{**}} \qquad \ldots(36)$$

where $S^{**} = S^{(1)}(q^{**}, p^{**}, \nu)$. Further taking into account the relation between the complex canonic variables with the real ones

$$q_j^{*} = \tilde{p}_j + i\,\tilde{q}_j, \quad p_j^{*} = \tilde{p}_j - i\,\tilde{q}_j$$

$$q_j^{*} = p_j^{*} + i\,q_j^{*}, \; p_j^{*} = p_j^{*} - i\,q_j^{*}$$

and denoting the function $S^{(1)}(p^* + i\,q^*, p^* - iq^*, \nu)$ by $W(q^*, p^*, \nu)$, we shall obtain by the formula (36),

$$\tilde{q}_j = q_j^{*} - \frac{1}{2i}\frac{\partial W}{\partial p_j^{*}}, \quad \tilde{p}_j = p_j^{*} + \frac{1}{2i}\frac{\partial W}{\partial q_j^{*}}. \qquad \ldots(37)$$

Comparing (35) and (37), we obtain $K^* = \frac{1}{2}w\ldots$ ...(38)

and the function $K = \sum k_{\nu\mu}\, q_1^{*\nu}\, q_2^{*\nu}\, p_1^{*\mu}\, p_2^{*\mu 2}$ will be realvalued. With the help of the formulae (4b), (19) and (33) its coefficients may be found which are given as follows.

$$\kappa_{2000} = -\frac{1}{2i}\left(-s_{2000}^{(1)} - s_{0020}^{(1)} + s_{1010}^{(1)}\right)$$

$$k_{0200} = \frac{1}{2i}\left(-s_{0200}^{(1)} - s_{0002}^{(1)} + s_{0101}^{(1)}\right)$$

$$k_{0020} = \frac{1}{2i}\left(s_{2000}^{(1)} + s_{0020}^{(1)} + s_{1010}^{(1)}\right)$$

$$k_{0002} = -\frac{1}{2i}\left(s_{0200}^{(1)} + s_{0002}^{(1)} + s_{0101}^{(1)}\right)$$

$$k_{1100} = \frac{1}{2i}\left(s_{1100}^{(1)} + s_{1001}^{(1)} + s_{0110}^{(1)} - s_{0011}^{(1)}\right)$$

$$k_{1010} = s_{2000}^{(1)} - s_{0020}^{(1)}$$

$$k_{1001} = \frac{1}{2}\left(s_{1100}^{(1)} + s_{1001}^{(1)} - s_{0110}^{(1)} - s_{0011}^{(1)}\right)$$

$$k_{0110} = \frac{1}{2}\left(s_{1100}^{(1)} - s_{1001}^{(1)} + s_{0110}^{(1)} - s_{0011}^{(1)}\right)$$

$$k_{0101} = s_{0200}^{(1)} - s_{0002}^{(1)}$$

$$k_{0011} = -\frac{1}{2i}\left(s_{1100}^{(1)} + s_{1001}^{(1)} + s_{0110}^{(1)} + s_{0011}^{(1)}\right). \qquad \ldots(93)$$

Thus the transfarmation of the Hamiltonian H_2 to the normal form given by (34) correct to the first order of eccentricity has been found. It is obtained through the transformations (16), (19) and (35) and the coefficients of the generating function K are given by (39).

5. Resonance Cases

Taking into view the complete study of the stability of the triangular libration points we shall need an investigation if resonances are present. Since we aim to apply *KAM*-theorem for stability which needs mainly the third and the fourth order terms, we shall only examine it the resonances of the third and the fourth order exist.

If a curve giving resonances be plotted for different values of e, then we shall need the value of λ_1 and λ_2 at least correct to $O\ (e^2)$, since $\lambda_1^{(1)} = \lambda_2^{(1)} = 0$. The quantities $\lambda_1^{(2)}$ and $\lambda_2^{(2)}$ are found by the periodicity conditions of the functions $s_{1010}^{(2)}$ and $s_{0101}^{(2)}$. For it let us take up the expansion (27) and here equating the coefficients with e^2, we shall get

$$\frac{ds_{1010}^{(2)}}{dv} = -2i\cos v\left(4a''_{0020}\ s_{2000}^{(1)} + a''_{1010}\ s_{1010}^{(1)} + a''_{1001}\ s_{1001}^{(1)} + a''_{0110}\ s_{1100}^{(1)}\right) + 2i\cos^2 v\, a''_{1010} + i\lambda_1^{(2)}$$

$$\frac{ds_{0101}^{(2)}}{dv} = -2i\cos v\left(4\,a''_{0002}\ s_{0200}^{(1)} + a''_{0110}\ s_{1001}^{(1)} + a''_{0101}\ s_{0101}^{(1)} + a''_{0011}\ s_{1100}^{(1)}\right) + 2i\cos^2 v\, a''_{0101} + i\ \lambda_2^{(2)}.$$

Substituting into the right-hand side expressions of these equations the value of the functions $s_{\mu\upsilon}^{(1)}$ given by (33) and choosing $\lambda_1^{(2)}$ and $\lambda_1^{(2)}$ such that the constant terms on the right-hand side are equal to zero (the condition of periodicity of $s_{1010}^{(2)}$ and $s_{0101}^{(2)}$) we shall obtain after some manipulations using the formulae (14b), (18) and the equation

$$\omega^4 - \omega^2 + 9\mu\,(1-\mu)\,b = 0$$

the following expressions for $\lambda_1^{(2)}$ and $\lambda_2^{(2)}$ and obtained.

$$\lambda_1^{(2)} = - \frac{\omega_1 \omega_2 (6 \omega_1^2 - 7)}{4 (4\omega_1^2 - 1) (2 \omega_1^2 - 1)},$$

$$\lambda_2^{(2)} = \frac{\omega_2 \omega_1^2 (6 \omega_2^2 - 7)}{4 (4 \omega_2^2 - 1) (2 \omega_2^2 - 1)} . \qquad ...(40)$$

These values coincide in forms with Markeev's[10] values.

Let the value of μ giving the resonance $k_1 \lambda_1 + k_2 \lambda_2 = N$ for small e and correct to $O(e^2)$ be given as $\mu = \mu^{(0)} + e^2 \mu^{(2)}$ where $\mu^{(0)}$ is the value of μ when $e = 0$ and $\mu^{(2)}$ is the contribution to the value of μ when $e \neq 0$ and it is considered correct to $O(e^2)$. Letting $\lambda_1 = \lambda_1 (\mu^{(0)} + e^2 \mu^{(2)})$ and $\lambda_2 = \lambda_2 (\mu^{(0)} + e^2 \mu^{(2)})$, we have on expansion by Taylor's theorem that

$$\lambda_1 = \lambda_1^{(0)} + e^2 \lambda_1^{(2)} + e^2 \mu^{(2)} \left(\frac{d\lambda_1}{d\mu} \right)_0$$

and similarly,

$$\lambda_2 = \lambda_2^{(0)} + e^2 \lambda_2^{(2)} + e^2 \mu^{(2)} \left(\frac{d\lambda_2}{d\mu} \right)_0$$

where $\lambda_1^{(2)}$ and $\lambda_2^{(2)}$ are given by (40) and the suffix (0) denotes that the value is to be taken when $\mu = \mu^{(0)}$. Putting these values in $k_1 \lambda_1 + k_2 \lambda_2 = N$ and equating the coefficient of e^2 to zero, we shall have

$$\mu^{(2)} = \frac{k_1 \lambda_1^{(2)} + k_2 \lambda_2^{(2)}}{k_2 \dfrac{d\omega_2}{d\mu} - k_1 \dfrac{d\omega_2}{d\mu}} .$$

The value of $\mu^{(2)}$ is calculated on putting $\mu = \mu^{(0)}$ on the right-hand side. The values of $\mu^{(0)}$ and $\mu^{(2)}$ for the different resonances of the third order are given in Tables II and III. In Table IV we have denoted by $\mu (e^2)$ the values of μ giving the limit for the linear stability for different values of b and e under the resonance case $3\lambda_2 = -1$. The values have been calculated on the formula $\mu = \mu^{(0)} + e^2 \mu^{(2)}$. e has been taken to vary from $e = 0.0$ to $e = 0.7$. We have here compared the values of $\mu (e)$ and $\mu (e^2)$ for the resonance under reference and we have examined if the resonance exists for a given value of b and e and if the study of stability will be necessary in the particular resonance case when higher order variational terms are taken into consideration. The conclusions are listed in the succeeding section.

We shall examine the following resonances of the third order if they exist under our range of linear stability given by Table 3.1 :

TABLE II

The value of $\mu^{(0)}$

b	$3\lambda_2 = -1$ $\mu^{(0)}$	$\lambda_1 + 2\lambda_2 = 0$ $\mu^{(0)}$	$2\lambda_1 + \lambda_2 = 1$ $\mu^{(0)}$	$\lambda_1 - 2\lambda_2 = 2$ $\mu^{(0)}$	$3\lambda_2 = -2$ $\mu^{(0)}$
.1	0.1255	0.23126	imaginary	imaginary	imaginary
.2	0.0583	0.09861	0.15071	0.15071	0.1641
.3	0 03803	0.06326	0.09421	0.09421	0.1018
.4	0.02823	0.04662	0.06872	0.06872	0.07407
.5	0.02245	0.03692	0.05413	0.0513	0.05826
.6	0.01864	0.03056	0.04466	0.04466	0.04303
.7	0.01593	0.02608	0.03802	0.03802	0.04086
.8	0.01391	0.02274	0.033095	0.033095	0.03556
.9	0.01235	0.02016	0.029303	0.029303	0.03147
1	0.0111	0.01811	0 026291	0.026291	0.02823
.75	0.014853	0 24294	0.0353854	0.0353854	0.038026

$\mu^{(0)}$ $\mu^{(0)} = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{8}{925b}}$ $\mu^{(0)} = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{4}{225b}}$ $\mu^{(0)} = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{16}{625b}}$

$\mu^{(0)} = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{16}{625b}}$ $\mu^{(0)} = \frac{1}{2} - \sqrt{\frac{1}{4} \sim \frac{20}{729b}}$

TABLE III

The values of $\mu^{(2)}$

b	$3\lambda_2 = -1$	$\lambda_1 + 2\lambda_2 = 0$	$2\lambda_1 + \lambda_2 = 1$	$\lambda_1 - 2\lambda_2 = 2$	$3\lambda_2 = -2$
.1	—0.83513	—3.8097591	imaginary	imaginary	imaginary
.2	—0.35404	—1.2733528	0.51445	1.5010664	0.568121
.3	—0.22567	—0.780191	0.295214	0.8613774	0.3198794
4	—0.165736	—0.56367	0.2083243	0.6078504	0.2242903
.5	—0.130984	—0.441488	0.1612059	0.470368	0.1730103
.6	—0.10829	—0.3629225	0.1315443	0.3838212	0.140912
.7	—0.09290	—0.308136	0.111132	0.3242611	0.1188955
8.	—0.080427	—0.267732	0.0962145	0.2807356	0.1028464
9.	—0.071262	—0.236704	0.084835	0.2475324	0.09621
1	—0.063972	—0.212128	0.075866	0.2213626	0.0809987

$\mu^{(2)}$ $\mu^{(2)} = -\frac{1596}{25515b\,(1-2\mu)}$ $\mu^{(2)} = -\frac{138}{675b\,(1-2\mu)}$ $\mu^{(2)} = \frac{0.0718769}{b\,(1-2\mu)}$

$\mu^{(2)} = \frac{0.209723}{b\,(1-2\mu)}$ $\mu^{(2)} = \frac{0.0764256}{b\,(1-2\mu)}$

(i) $3\lambda_2 = -1$, (ii) $\lambda_1 + 2\lambda_2 = 0$, (iii) $2\lambda_2 + \lambda_2 = 1$,
(iv) $\lambda_1 - 2\lambda_2 = 2$, (v) $3\lambda_2 = -2$.

The corresponding values are given by Tables IV—VIII.

Coming to resonances of the fourth order we may proceed similarly and it may be seen that the study of the stability will be needed for the following resonance.

(i) $4\lambda_2 = -1$, (ii) $\lambda_1 + 3\lambda_2 = 0$, (iii) $\lambda_1 - 3\lambda_2 = 2$,
(iv) $2(\lambda_1 + \lambda_2) = 1$, (v) $3\lambda_1 + \lambda_2 = 2$, (vi) $3\lambda_1 - \lambda_2 = 3$,
(vii) $\lambda_1 + 3\lambda_2 = -1$, (viii) $4\lambda_1 = 3$.

TABLE IV

The values for μ (e^2)

$(3\lambda_2 = -1)$

b	$e = 00$	$e = 0.1$	$e = 0.2$	$e = 0.3$	$e = 0.4$	$e = 0.5$	$e = 0.6$	$e = 0.7$
0.1	0.1255	0.117	0.092	0.05	negative	—ve	—ve	—ve
0.2	0.058	0.055	0.044	0.026	0.002	—ve	—ve	—ve
0.3	0.038	0.036	0.029	0.018	0.002	—ve	—ve	—ve
0.4	0.028	0.027	0.022	0.013	0.002	—ve	—ve	—ve
0.5	0.022	0.021	0.017	0.011	0.001	—ve	—ve	—ve
0.6	0.019	0.018	0.014	0.009	0.001	—ve	—ve	—ve
0.7	0.016	0.015	0.012	3.008	0.001	—ve	—ve	—ve
0.8	0.014	0.013	0.011	0.007	0.001	—ve	—ve	—ve
0.9	0.012	0.012	0.009	0.006	0.001	—ve	—ve	—ve
1.0	0.012	0.010	0.009	0.005	0.001	—ve	—ve	—ve

—ve = negative

TABLE V

The values for μ (e^2)

$(\lambda_1 + 2\lambda_2 = 0)$

b	$e = 0.0$	$e = 0.1$	$c = 0.2$	$e = 0.3$	$e = 0.4$	$e = 0.5$	$e = 0.6$	$e = 0.8$
0.1	0.231	0.193	0.079	—ve	—ve	—ve	—ve	—ve
0.2	0.099	0.0859	0.048	—ve	—ve	—ve	—ve	—ve
0.3	0.063	0.055	0.032	—ve	—ve	—ve	—ve	—ve
0.4	0.047	0.041	0.024	—ve	—ve	—ve	—ve	—ve
0.5	0.037	0.033	0.019	—ve	—ve	—ve	—ve	—ve
0.6	0.031	0.027	0.016	—ve	—ve	—ve	—ve	—ve
0.7	0.026	0.023	0.014	—ve	—ve	—ve	—ve	—ve
0.8	0.023	0.020	0.012	—ve	—ve	—ve	—ve	—ve
0.9	0.020	0.018	0.017	—ve	—ve	—ve	—ve	—ve
1.0	0.018	0.016	0.001	—ve	—ve	—ve	—ve	—ve

As shown by Moser[2], the motion under the resonances $\lambda_1 - 3\lambda_2 = 2$ and $3\lambda_1 - \lambda_2 = 3$ cannot lead to instability and the cases $\lambda_1 + 3\lambda_2 = 0$ and $\lambda_1 + 3\lambda_2 = -1$ reduce to the case $\lambda_1 + 3\lambda_2 = m$ (an integer). Thus it remains to study the stability only for the following five cases :

$$\text{(i) } 4\lambda_2 = m, \quad \text{(ii) } \lambda_1 + 3\lambda_2 = m, \quad \text{(iii) } 2(\lambda_1 + \lambda_2) = m,$$

$$\text{(iv) } 3\lambda_1 + \lambda_2 = m, \quad \text{(v) } 4\lambda_1 = m,$$

where m is an integer.

TABLE VI

The values for $\mu\ (e^2)$

$(2\lambda_1 + \lambda_2 = 1)$

b	$e = 0.0$	$e = 0.1$	$e = 0.2$	$e = 0.3$	$e = 0.4$	$e = 0.5$	$e = 0.6$	$e = 0.7$
0.1	Imag.	Imag.	Imog.	Imag.	Imag.	Imag.	Imag.	Imag.
0.2	0.151	0.156	0.017	0.197	0.233	0.279	0.339	0.403
0.3	0.094	0.097	0.106	0.121	0.141	0.168	0.200	0.239
0.4	0.069	0.071	0.077	0.087	0.102	0.121	0.144	0.171
0.5	0.054	0.056	0.061	0.686	0.080	0.094	0.112	0.133
0.6	0.045	0 046	0.050	0.066	0.066	0.078	0.092	0.109
0.7	0.038	0.039	0.042	0.048	0.056	0.066	0.078	0.092
0.8	0.033	0.034	0.037	0.042	0.048	0.057	0.068	0.081
0.9	0.029	0.030	0.033	0.037	0.043	0.051	0.060	0.070
1.0	0.026	0.027	0.029	0.033	0.036	0.045	0.054	0.063

Imag. = Imaginary

TABLE VII

The values for $\mu\ (e^2)$

$(\lambda_1 - 2\lambda_2 = 2)$

b	$e = 0.0$	$e = 0.1$	$e = 0.2$	$e = 0.3$	$e = 0.4$	$e = 0.5$	$e = 0.6$	$e = 0.7$
0.1	Imag.	Imag.	Imag.	Imag.	Imag.	Imag.	Imag.	Imag.
0.2	0.151	0.166	0.211	0.286	0.391	0.526	0.691	0.886
0.3	0.094	0.103	0.129	0.172	0.232	0.340	0.404	0.516
0.4	0.069	0.0748	0.093	0.123	0.166	0.221	0.288	0.367
0.5	0.054	0.0588	0.0729	0.0965	0.129	0.1717	0.2235	0.2846
0.6	0.045	0.048	0.060	0.079	0.106	0.141	0.183	0.233
0.7	0.038	0.0413	0.051	0.0672	0.09	0.1191	0.1548	0.1969
0.8	0.033	0.0368	0.0443	0.0623	0.085	0.1142	0.1498	0.192
0.9	0.029	0.032	0.0392	0.0515	0 0689	0.0912	0.1184	0.1506
1.0	0.026	0.0285	0.035	0.046	0.062	0.082	0.106	0.1348

Imag. = Imaginary

TABLE VIII

The values for u (e^2)

(3 = −2)

b	$e = 0.0$	$e = 0.1$	$e = 0.2$	$e = 0.3$	$e = 0.4$	$e = 0.5$	$e = 0.6$	$e = 0.7$
0.1	Imag.	Imag.	Imag.	Imag.	Imag.	Imag.	Imag.	Imag.
0.2	0.104	0.170	0.187	0.215	0.255	0.306	0.369	0.443
0.3	0.102	0.105	0.115	0.131	0.153	0.182	0.217	0.259
0.4	0.074	0.077	0.083	0.094	0.110	0.130	0.155	0.184
0.5	0.058	0.060	0.065	0.074	0.086	0.102	0.121	0.143
0.6	0.048	0.049	0.054	0.061	0.071	0.083	0.092	0.117
0.7	0.041	0.042	0.046	0.052	0.060	0.071	0.084	0.099
0.8	0.036	0.037	0.040	0.045	0.052	0.061	0.072	0.086
0.9	0.031	0.032	0.035	0.040	0.047	0.056	0.066	0.079
1.0	0.028	0.029	0.331	0.036	0.041	0.048	0.057	0.068

Imag. = Imaginary

6. CONCLUSIONS

In section 3 we found out the characteristic exponents correct to $O(e^2)$ and we have calculated the values of μ giving the range of linear stability for different values of e. These values denoted by $\mu(e)$ have been given by Table I. In section 4 we have normalized the second order terms by Birkhoff's transformations and also we have calculated the various terms as resulting after transformations. In section 5 we have found the values of μ corresponding to different types of resonances. e has been taken to vary from 0.0 to 0.7 and b from 0.1 to 1.0. From Table IV we find that the resonances of the type of $3\lambda_2 = -1$ exists for $e < 0.5$ and in all such cases the motion is stable in the linear sense since the corresponding values of $\mu(e)$ is greater than $\mu(e^2)$. So for the investigation of non-linear stability this type of resonance case has to be taken into account. The negative values of $\mu(e^2)$ indicate that the resonance case will not exist for our range of values of $\mu(e)$ under consideration.

Table V shows the resonance case $\lambda_1 + 2\lambda_2 = 0$ exists for $b = 0.1$ only for values of e such that $0.1 < e < 0.3$, and for $b \geqslant 0.2$ it exists for $e \leqslant 0.2$. Since the values of $\mu(e^2)$ corresponding to the resonance are all less than the value $\mu(e)$ corresponding to the range of linear stability, for non-linear stability the study of the resonance case will be necessary.

Table VI shows that the resonance case $2\lambda_1 + \lambda_2 = 1$ will not exist for $b = 0.1$ and for $b \geqslant 0.2$, the resonance exists for all e from 0.0 to 0.7, but the the linear stability is possible only for $e < 0.2$, and for $e \geqslant 0.2$ the motion will be unstable since even the linear stability does not hold.

Table VII shows that for $b = 0.1$ the resonance $\lambda_1 - 2\lambda_2 = 2$ does not exist but for values of b it does exist. The values of $\mu\,(e^2)$ are greater than the corresponding values $\mu\,(e)$ for the linear stability. It shows that the motion can be stable only for the values of e such that $e < 0.1$, and for $e \geqslant 0.1$ the motion will be unstable.

Table VIII shows that in the resonance case $3\lambda_2 = -2$, the motion can be stable only for $e < 0.1$, and for $e \geqslant 0.1$ the motion will be unstable since $\mu\,(e) < \mu\,(e^2)$ for all b and all e.

Similarly resonances of the fourth order will exist for some values of b and e and for non-linear stability these cases have to be taken into account. We have not calculated the corresponding values of $\mu\,(e^2)$, but it is guessed that the resonances of the fourth type will exist, similar to the classical case as examined by Markeev[10].

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COMMENTS ON "STEADY PLANE MHD FLOWS WITH CONSTANT SPEED ALONG EACH STREAMLINE"

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A paper entitled "Steady Plane MHD Flows with Constant Speed Along Each Streamline" authored by M. A. Sattar was published in *Indian J. pure appl. Math.* 18 (1987), 548-56. Exactly the same work authored by M. A. Sattar and O. P. Chandna appeared in *J. Math. Phy. Sci.* **22** (1988), 321-33. Although Dr Sattar did this work jointly with me, I was totally unaware of its submission to these journals by him. The mathematical analysis of this work has two cases when

$$J = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

is nowhere zero in the region of flow and when J is everywhere zero in the region of flow. The summing up of the case when $J = 0$ everywhere needs correcting. This summing up should read : The steady plane flow of a viscous incompressible fluid of infinite electrical conductivity having constant velocity magnitude along each streamline has only constant solutions given by (3.31) when $J = 0$ and the streamlines for such a flow are parallel straight lines.

The above correction implies that the non-constant solutions given in (3.35) are not possible for the subcase taken. This subcase has

$$u^2 + \phi^2(u) = u^2 + v^2 = \text{constant} = c\text{-say}$$

$$u_x + \phi'(u)\, u_y = 0 \qquad \ldots(3.27)$$

$$\{u^2 + u\phi(u)\,\phi'(u)\}\, u_x + \{u\phi(u) + \phi^2(u)\,\phi'(u)\}\, u_y = 0 \qquad \ldots(3.28)$$

$$\{2u\,\phi(u) + \phi^2(u)\,\phi'(u) - u^2\phi'(u)\}\, u_x + \{\phi^2(u) - u^2 - 2u\phi(u)\,\phi'(u)\}\, u_y = 0 \qquad \ldots(3.29)$$

where c is an arbitrary constant and $\dfrac{\partial u}{\partial y} \neq 0$.

Since $2u+2\phi(u)\,\phi'(u)=0$ is a consequence of the assumption $u^2 + \phi^2(u) =$ constant taken as Case III in paper, it follows that equation (3.28) is identically satisfied. Eliminating $\dfrac{\partial u}{\partial x}$ from equations (3.27) and (3.29) and requiring $\dfrac{\partial u}{\partial y} \neq 0$, we find that $\phi(u)$ must satisfy

$$\phi^2(u)\,\phi'^2(u) - u^2\,\phi'^2(u) - \phi^2(u) + u^2 + 4u\phi(u)\,\phi'(u) = 0.$$

Employing $\phi(u) = \pm\sqrt{c-u^2}$ in this equation to be satisfied by $\phi(u)$, we conclude that

$$c = 0$$

and, therefore,

$$u^2 + \phi^2(u) = u^2 + v^2 = 0.$$

This equation implies that subcase III, for the case $J = 0$, given in the paper is not physically possible.

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